Appendix D

$U(2\Omega) \supset [U(\Omega) \supset SO(\Omega)] \otimes SU(2)$ pairing symmetry

With the $4\Omega^2$ number of one-body operators $u^r_\mu(i,j)$; $r = 0,1$, defined in Sec. 3.2, generating $U(2\Omega)$ algebra, it is easily seen that the operators $u^0(i,j)$, which are Ω^2 in number, generate $U(\Omega)$ algebra. Similarly the operators $C(i,j) = u^0(i,j) - u^0(j,i)$, $i > j$, which are $\Omega(\Omega-1)/2$ in number, generate the $SO(\Omega)$ sub-algebra of $U(\Omega)$. The spin operator $\hat{S} = S^1_\mu$, the number operator $\hat{n}$ and the quadratic Casimir operators C_2's of $U(\Omega)$ and $SO(\Omega)$ are

$$
\begin{gathered}
S^1_\mu = \frac{1}{\sqrt{2}} \sum_{i=1}^{\Omega} u^1_\mu(i,i)\,, \\
\hat{n} = \sum_i n_i\,, \quad n_i = \sqrt{2} u^0(i,i)\,, \\
C_2(U(\Omega)) = 2 \sum_{i,j} u^0(i,j) u^0(j,i)\,, \\
C_2(SO(\Omega)) = 2 \sum_{i>j} C(i,j) C(j,i)\,.
\end{gathered} \tag{D1}
$$

The structure of $C_2(U(\Omega))$ in terms of the number operator and the $\hat{S}\cdot\hat{S} = \hat{S}^2$ operator is,

$$
\begin{aligned}
C_2(U(\Omega)) &= \hat{n}\left(\Omega + 2 - \tfrac{\hat{n}}{2}\right) - 2\hat{S}^2\,, \\
\langle C_2(U(\Omega))\rangle^{m,S} &= m\left(\Omega + 2 - \tfrac{m}{2}\right) - 2S(S+1)\,.
\end{aligned} \tag{D2}
$$

Note that $\langle C_2(U(\Omega))\rangle^{\{f\}} = \sum_i f_i(f_i + \Omega + 1 - 2i)$. As $U(2\Omega) \supset U(\Omega) \otimes SU(2)$ with the $SU(2)$ algebra generating total spin S, the $U(\Omega)$ irreps are labeled by two column irreps $\{2^p 1^q\}$ with $m = 2p + q$, $S = q/2$. As a consequence, the $SO(\Omega)$ irreps are also of two column type and we will denote them by $[2^{v_1} 1^{v_2}]$. Here, $v_S = 2v_1 + v_2$ is called seniority and $\tilde{s} = v_2/2$ is called reduced spin. We also have [Wy-74]

$$
\begin{aligned}
\langle C_2(SO(\Omega))\rangle^{\langle\omega\rangle} &= \sum_i \omega_i(\omega_i + \Omega - 2i) \\
\Rightarrow \langle C_2(SO(\Omega))\rangle^{\langle 2^{v_1} 1^{v_2}\rangle} &= v_S\left(\Omega + 1 - \frac{v_S}{2}\right) - 2\tilde{s}(\tilde{s}+1)\,.
\end{aligned}
\tag{D3}
$$

After some commutator algebra it can be shown that,

$$
\begin{aligned}
&2H_p = -C_2(SO(\Omega)) + \hat{n}\left(\Omega + 1 - \tfrac{\hat{n}}{2}\right) - 2\hat{S}^2\,, \\
&\left\langle H_p\right\rangle^{(m,S,v_S,\tilde{s})} = \tfrac{1}{4}(m - v_S)(2\Omega + 2 - m - v_S) + [\tilde{s}(\tilde{s}+1) - S(S+1)]\,,
\end{aligned}
\tag{D4}
$$

where the pairing Hamiltonian H_p is defined by Eq. (3.2.2). Classification of $U(2\Omega) \supset [U(\Omega) \supset SO(\Omega)] \otimes SU(2)$ states defined by $(m, S, v_S, \tilde{s})$ quantum numbers is needed, i.e., $(m, S) \to (v_S, \tilde{s})$ reductions are required and they are obtained by group theory. Using the tabulations in [Wy-70], results are given in Tables D.1 and D.2 for: (i) $m \leq 4$, $\Omega \geq 4$ and (ii) $m = 6$, $\Omega = 6$ and $m = 5-8$, $\Omega = 8$, respectively.

Table D.1: $(m,S) \to (v_S,\tilde{s})$ reductions for $m \leq 4$ and $\Omega \geq 4$.

(m,S)	$(v_S,\tilde{s})$
$(0,0)$	$(0,0)$
$(1,\frac{1}{2})$	$(1,\frac{1}{2})$
$(2,0)$	$(2,0)$, $(0,0)$
$(2,1)$	$(2,1)$
$(3,\frac{1}{2})$	$(3,\frac{1}{2})$, $(1,\frac{1}{2})$
$(3,\frac{3}{2})$	$\{(1,\frac{1}{2})_{\Omega=4};\ (2,1)_{\Omega=5};\ (3,\frac{3}{2})_{\Omega\geq 6}\}$
$(4,0)$	$(4,0)$, $(2,0)$, $(0,0)$
$(4,1)$	$\{(2,0)_{\Omega=4};\ (3,\frac{1}{2})_{\Omega=5};\ (4,1)_{\Omega\geq 6}\}$, $(2,1)$
$(4,2)$	$\{(0,0)_{\Omega=4};\ (1,\frac{1}{2})_{\Omega=5};\ (2,1)_{\Omega=6},\ (3,\frac{3}{2})_{\Omega=7};\ (4,2)_{\Omega\geq 8}\}$

Table D.2: $(m,S) \to (v_S,\tilde{s})$ irrep reductions for $(\Omega = 6; m = 6)$ and $(\Omega = 8; m = 5-8)$. Note that the dimensions $d_f(\Omega,m,S)$ of the (m,S) and $d(\Omega,v_S,\tilde{s})$ of the $(v_S,\tilde{s})$ space are given as subscripts; $d_f(\Omega,m,S) = \sum_{v_S,\tilde{s}} d(\Omega,v_S,\tilde{s})$.

Ω	$(m,S)_{d_f(\Omega,m,S)}$	$(v_S,\tilde{s})_{d(\Omega,v_S,\tilde{s})}$
6	$(6,0)_{175}$	$(6,0)_{70}$, $(4,0)_{84}$, $(2,0)_{20}$, $(0,0)_1$
	$(6,1)_{189}$	$(4,1)_{90}$, $(4,0)_{84}$, $(2,1)_{15}$
	$(6,2)_{35}$	$(2,1)_{15}$, $(2,0)_{20}$
	$(6,3)_1$	$(0,0)_1$
8	$(5,\frac{1}{2})_{1008}$	$(5,\frac{1}{2})_{840}$, $(3,\frac{1}{2})_{160}$, $(1,\frac{1}{2})_8$
	$(5,\frac{3}{2})_{504}$	$(5,\frac{3}{2})_{448}$, $(3,\frac{3}{2})_{56}$
	$(5,\frac{5}{2})_{56}$	$(3,\frac{3}{2})_{56}$
	$(6,0)_{1176}$	$(6,0)_{840}$, $(4,0)_{300}$, $(2,0)_{35}$, $(0,0)_1$
	$(6,1)_{1512}$	$(6,1)_{1134}$, $(4,1)_{350}$, $(2,1)_{28}$
	$(6,2)_{420}$	$(4,1)_{350}$, $(4,2)_{70}$
	$(6,3)_{28}$	$(2,1)_{28}$
	$(7,\frac{1}{2})_{2352}$	$(7,\frac{1}{2})_{1344}$, $(5,\frac{1}{2})_{840}$, $(3,\frac{1}{2})_{160}$, $(1,\frac{1}{2})_8$
	$(7,\frac{3}{2})_{1344}$	$(5,\frac{1}{2})_{840}$, $(5,\frac{3}{2})_{448}$, $(3,\frac{3}{2})_{56}$
	$(7,\frac{5}{2})_{216}$	$(3,\frac{1}{2})_{160}$, $(3,\frac{3}{2})_{56}$
	$(7,\frac{7}{2})_8$	$(1,\frac{1}{2})_8$
	$(8,0)_{1764}$	$(8,0)_{588}$, $(6,0)_{840}$, $(4,0)_{300}$, $(2,0)_{35}$, $(0,0)_1$
	$(8,1)_{2352}$	$(6,0)_{840}$, $(6,1)_{1134}$, $(4,1)_{350}$, $(2,1)_{28}$
	$(8,2)_{720}$	$(4,0)_{300}$, $(4,1)_{350}$, $(4,2)_{70}$
	$(8,3)_{63}$	$(2,0)_{35}$, $(2,1)_{28}$
	$(8,4)_1$	$(0,0)_1$