# Appendix F

# Excess parameter $\gamma_2(m, f_m)$ in terms of $SU(\Omega)$ Racah coefficients

The formula for $\gamma_2(m, f_m)$, given by Eq. (4.4.7), involves $\overline{\langle H^4 \rangle^{m,f_m}}$. As the Hamiltonian in Eq. (4.3.1) is a direct sum of matrices in $f_2 = \{2\}$ and $\{1^2\}$ spaces, we have

$$\overline{\langle H^4 \rangle^{m,f_m}} = \overline{\langle (H_{\{2\}} + H_{\{1^2\}})^4 \rangle^{m,f_m}} . \tag{F1}$$

Expanding the RHS of Eq. (F1) using the cyclic invariance of the averages and applying the property that terms with odd powers of $H_{\{2\}}$ and $H_{\{1^2\}}$ will vanish [see Eq. (4.3.6)], we have

$$\begin{aligned} \overline{\langle H^4 \rangle^{m,f_m}} &= \overline{\langle (H_{\{2\}})^4 \rangle^{m,f_m}} + \overline{\langle (H_{\{1^2\}})^4 \rangle^{m,f_m}} + 4\overline{\langle (H_{\{2\}})^2 (H_{\{1^2\}})^2 \rangle^{m,f_m}} \\ &+ 2\overline{\langle H_{\{2\}} H_{\{1^2\}} H_{\{2\}} H_{\{1^2\}} \rangle^{m,f_m}} . \end{aligned} \tag{F2}$$

Writing $H$ in terms of the unit tensors $B$'s using Eq. (4.4.3), the first two terms in Eq. (F2) will give

$$\overline{\left\langle H_{f_2}^4 \right\rangle^{m,f_m}}$$

$$= \frac{1}{d_\Omega(f_m)} \sum_{\nu_1,\nu_2,\nu_3,\nu_4,\boldsymbol{F}_{\nu_1},\boldsymbol{F}_{\nu_2},\boldsymbol{F}_{\nu_3},\boldsymbol{F}_{\nu_4},\omega_{\nu_1},\omega_{\nu_2},\omega_{\nu_3},\omega_{\nu_4}} \langle f_m v_1 \mid B(f_2 \boldsymbol{F}_{\nu_1} \omega_{\nu_1}) \mid f_m v_2 \rangle$$

$$\times \langle f_m v_2 \mid B(f_2 \boldsymbol{F}_{v_2} \omega_{v_2}) \mid f_m v_3 \rangle \langle f_m v_3 \mid B(f_2 \boldsymbol{F}_{v_3} \omega_{v_3}) \mid f_m v_4 \rangle$$

$$\times \langle f_m v_4 \mid B(f_2 \boldsymbol{F}_{v_4} \omega_{v_4}) \mid f_m v_1 \rangle \tag{F3}$$

$$\times \overline{W(f_2 \boldsymbol{F}_{v_1} \omega_{v_1}) W(f_2 \boldsymbol{F}_{v_2} \omega_{v_2}) W(f_2 \boldsymbol{F}_{v_3} \omega_{v_3}) W(f_2 \boldsymbol{F}_{v_4} \omega_{v_4})}\,.$$

Using Eq. (4.4.5), it is easy to see that the term $\overline{\left\langle H^4_{f_2} \right\rangle^{m,f_m}}$ will have non-zero contribution in three cases, (i) $\delta_{\boldsymbol{F}_{v_1},\boldsymbol{F}_{v_2}} = 1$, $\delta_{\omega_{v_1},\omega_{v_2}} = 1$, $\delta_{\boldsymbol{F}_{v_3},\boldsymbol{F}_{v_4}} = 1$, $\delta_{\omega_{v_3},\omega_{v_4}} = 1$; (ii) $\delta_{\boldsymbol{F}_{v_1},\boldsymbol{F}_{v_4}} = 1$, $\delta_{\omega_{v_1},\omega_{v_4}} = 1$, $\delta_{\boldsymbol{F}_{v_2},\boldsymbol{F}_{v_3}} = 1$, $\delta_{\omega_{v_2},\omega_{v_3}} = 1$; and (iii) $\delta_{\boldsymbol{F}_{v_1},\boldsymbol{F}_{v_3}} = 1$, $\delta_{\omega_{v_1},\omega_{v_3}} = 1$, $\delta_{\boldsymbol{F}_{v_2},\boldsymbol{F}_{v_4}} = 1$, $\delta_{\omega_{v_2},\omega_{v_4}} = 1$. The first two cases are equivalent due to cyclic invariance of the traces and they can be called direct terms whereas the third case involves cross-correlations and thus is called the exchange term. For (i) and (ii), applying the Wigner-Eckart theorem and carrying out simplifications using the properties of the Wigner coefficients (see Appendix E), the direct terms reduce to $2\left[\overline{\left\langle H^2_{f_2} \right\rangle^{m,f_m}}\right]^2$. Similarly, for the exchange term, reordering of the Wigner coefficients [see Eq. (E7)] yields an expression in terms of a new Racah coefficient. With these, we have

$$\overline{\left\langle H^4_{f_2} \right\rangle^{m,f_m}} = 2\left[\overline{\left\langle H^2_{f_2} \right\rangle^{m,f_m}}\right]^2 + \lambda^4_{f_2} \left[d_4(F_2)\right]^2 d_\Omega(f_m)$$

$$\times \sum_{\boldsymbol{F}_{v_1},\boldsymbol{F}_{v_2},\rho_1,\rho_2,\rho_3,\rho_4} \frac{1}{\sqrt{d_\Omega(\boldsymbol{F}_{v_1}) d_\Omega(\boldsymbol{F}_{v_2})}} U(f_m \overline{f_m} f_m f_m; (\boldsymbol{F}_{v_1})_{\rho_1\rho_3} (\boldsymbol{F}_{v_2})_{\rho_2\rho_4})$$

$$\times \langle f_m \,||\, B(f_2 \boldsymbol{F}_{v_1}) \,||\, f_m \rangle_{\rho_1} \langle f_m \,||\, B(f_2 \boldsymbol{F}_{v_2}) \,||\, f_m \rangle_{\rho_2} \tag{F4}$$

$$\times \langle f_m \,||\, B(f_2 \boldsymbol{F}_{v_1}) \,||\, f_m \rangle_{\rho_3} \langle f_m \,||\, B(f_2 \boldsymbol{F}_{v_2}) \,||\, f_m \rangle_{\rho_4}\,.$$

In Eq. (F4), the multiplicity labels appearing in the new $U$-coefficient [this is quite different from the $U$-coefficient appearing in Eq. (4.4.10)] can be easily understood from the corresponding labels in the reduced matrix elements. Similarly, we have

$$\overline{\left\langle H^2_{\{2\}} H^2_{\{1^2\}} \right\rangle^{m,f_m}} = \left\{\overline{\left\langle H^2_{\{2\}} \right\rangle^{m,f_m}}\right\} \left\{\overline{\left\langle H^2_{\{1^2\}} \right\rangle^{m,f_m}}\right\}, \tag{F5a}$$

$$\overline{\langle H_{\{2\}} H_{\{1^2\}} H_{\{2\}} H_{\{1^2\}} \rangle^{m,f_m}} = \lambda_{\{2\}}^2 \lambda_{\{1^2\}}^2 d_4(\{2\}) d_4(\{1^2\}) d_\Omega(f_m)$$

$$\times \sum_{\boldsymbol{F}_{\nu_1}, \boldsymbol{F}_{\nu_2}, \rho_1, \rho_2, \rho_3, \rho_4} \frac{1}{\sqrt{d_\Omega(\boldsymbol{F}_{\nu_1}) d_\Omega(\boldsymbol{F}_{\nu_2})}} U(f_m \overline{f_m} f_m f_m; (\boldsymbol{F}_{\nu_1})_{\rho_1 \rho_3} (\boldsymbol{F}_{\nu_2})_{\rho_2 \rho_4}) \tag{F5b}$$

$$\times \langle f_m \,||\, B(\{2\}\boldsymbol{F}_{\nu_1}) \,||\, f_m \rangle_{\rho_1} \langle f_m \,||\, B(\{1^2\}\boldsymbol{F}_{\nu_2}) \,||\, f_m \rangle_{\rho_2}$$

$$\times \langle f_m \,||\, B(\{2\}\boldsymbol{F}_{\nu_1}) \,||\, f_m \rangle_{\rho_3} \langle f_m \,||\, B(\{1^2\}\boldsymbol{F}_{\nu_2}) \,||\, f_m \rangle_{\rho_4} .$$

Substituting the results in Eqs. (F4), (F5a) and (F5b) in Eq. (F2) gives $\overline{\langle H^4 \rangle^{m,f_m}}$. Using this and Eqs. (4.5.5) and (4.4.7), we have the analytical result for the excess parameter $\gamma_2(m, f_m)$. This involves $SU(\Omega)$ Racah coefficients with multiplicity labels and evaluation of these is in general complicated [Gl-05, Kl-09]. Similarly, evaluation of the reduced matrix elements in Eq. (F4) is also complicated. The only simple situation is, when the multiplicity labels are all unity. We denote the $U(\Omega)$ irreps that satisfy this as $f_m^{(g)}$ and we have verified that one of these irreps is $\{4^r\}$ where $m = 4r$. For these irreps, the expression for $\gamma_2$ is,

$$\left[\gamma_2(m, f_m^{(g)}) + 1\right] = \left[\overline{\langle H^2 \rangle^{m,f_m^{(g)}}}\right]^{-2}$$

$$\times \left\{ \sum_{f_a, f_b = \{2\}, \{1^2\}} \frac{\lambda_{f_a}^2 \lambda_{f_b}^2}{d_\Omega(f_a) d_\Omega(f_b)} \sum_{\boldsymbol{F}_{\nu_1}, \boldsymbol{F}_{\nu_2}} \frac{d_\Omega(f_m^{(g)})}{\sqrt{d_\Omega(\boldsymbol{F}_{\nu_1}) d_\Omega(\boldsymbol{F}_{\nu_2})}} \right. \tag{F6}$$

$$\left. \times\, U(f_m^{(g)} \overline{f_m^{(g)}} f_m^{(g)} f_m^{(g)}; \boldsymbol{F}_{\nu_1} \boldsymbol{F}_{\nu_2})\, \mathscr{Q}^{\nu_1}(f_a : m, f_m^{(g)})\, \mathscr{Q}^{\nu_2}(f_b : m, f_m^{(g)}) \right\} .$$

The $\mathscr{Q}^{\nu}(f_2 : m, f_m)$ in Eq. (F6) are defined by Eq. (4.5.6). They can be calculated using $X_{UU}$ given in Table 4.4. Therefore the only unknown in Eq. (F6) is the $SU(\Omega)$ Racah coefficient $U(f_m^{(g)} \overline{f_m^{(g)}} f_m^{(g)} f_m^{(g)}; \boldsymbol{F}_{\nu_1} \boldsymbol{F}_{\nu_2})$. There are many attempts in the past to derive analytical formulation and also to develop numerical methods for evaluating general $SU(N)$ Racah coefficients [Bi-68, Lo-70a, Lo-70, Bl-87, Bi-82, Se-88, Vi-95]. There are also attempts to derive analytical formulas for some simple class of Racah coefficients; see [Vi-95, Li-90] and references therein. In addition, there is a recent effort to develop a new numerical method for evaluating $SU(N)$ Racah coefficients with multiplicities [Gl-05, Kl-09]. From all the attempts we made in trying to use these results, we conclude that further group theoretical work on $SU(N)$ Racah coefficients is needed to be able to derive analytical formulas for, or for evaluating numerically, the Racah coefficients appearing in Eq. (F6).