Appendix H

Basic binary correlation results

We denote a k_H-body operator as,

$$H(k_H) = \sum_{\alpha,\beta} v_H^{\alpha\beta}\, \alpha^\dagger(k_H)\beta(k_H)\,. \tag{H1}$$

Here, $\alpha^\dagger(k_H)$ is the k_H particle creation operator and $\beta(k_H)$ is the k_H particle annihilation operator. Similarly, $v_H^{\alpha\beta}$ are matrix elements of the operator H in k_H particle space i.e., $v_H^{\alpha\beta} = \langle k_H\beta \mid H \mid k_H\alpha\rangle$ (it should be noted that Mon and French [Mo-73, Mo-75] used operators with daggers to denote annihilation operators and operators without daggers to denote creation operators). Following basic traces will be used throughout,

$$\sum_\alpha \alpha^\dagger(k)\alpha(k) = \binom{\hat{n}}{k} \quad \Rightarrow \left\langle \sum_\alpha \alpha^\dagger(k)\alpha(k) \right\rangle^m = \binom{m}{k}. \tag{H2}$$

$$\sum_\alpha \alpha(k)\alpha^\dagger(k) = \binom{N-\hat{n}}{k} \quad \Rightarrow \left\langle \sum_\alpha \alpha(k)\alpha^\dagger(k) \right\rangle^m = \binom{\widetilde{m}}{k};\quad \widetilde{m} = N-m\,. \tag{H3}$$

$$\begin{gathered}\sum_\alpha \alpha^\dagger(k)B(k')\alpha(k) = \binom{\hat{n}-k'}{k}B(k') \\ \Rightarrow \left\langle \sum_\alpha \alpha^\dagger(k)B(k')\alpha(k) \right\rangle^m = \binom{m-k'}{k}B(k')\,.\end{gathered} \tag{H4}$$

$$\sum_{\alpha} \alpha(k) B(k') \alpha^{\dagger}(k) = \binom{N-\hat{n}-k'}{k} B(k')$$

$$\Rightarrow \left\langle \sum_{\alpha} \alpha(k) B(k') \alpha^{\dagger}(k) \right\rangle^{m} = \binom{\tilde{m}-k'}{k} B(k') . \tag{H5}$$

Equation (H2) follows from the fact that the average should be zero for $m < k$ and one for $m = k$ and similarly, Eq. (H3) follows from the same argument except that the particles are replaced by holes. Equation (H4) follows first by writing the k'-body operator $B(k')$ in operator form using Eq. (H1), i.e.,

$$B(k') = \sum_{\beta,\gamma} v_B^{\beta\gamma} \beta^{\dagger}(k') \gamma(k') , \tag{H6}$$

and then applying the commutation relations for the fermion creation and annihilation operators. This gives $\sum_{\beta,\gamma} v_B^{\beta\gamma} \beta^{\dagger}(k') \sum_{\alpha} \alpha^{\dagger}(k) \alpha(k) \gamma(k')$. Now applying Eq. (H2) to the sum involving α gives Eq. (H4). Eq. (H5) follows from the same arguments except one has to assume that $B(k')$ is fully irreducible $\nu = k'$ operator and therefore, it has particle-hole symmetry. For a general $B(k')$ operator, this is valid only in the $N \to \infty$ limit. Therefore, this equation has to be applied with caution.

Using the definition of the H operator in Eq. (H1), we have

$$\begin{aligned} \overline{\langle H(k_H) H(k_H) \rangle^{m}} &= \sum_{\alpha,\beta} \overline{\left\{ v_H^{\alpha\beta} \right\}^2} \left\langle \alpha^{\dagger}(k_H) \beta(k_H) \beta^{\dagger}(k_H) \alpha(k_H) \right\rangle^{m} \\ &= v_H^2 \left\langle \sum_{\alpha} \alpha^{\dagger}(k_H) \left\{ \sum_{\beta} \beta(k_H) \beta^{\dagger}(k_H) \right\} \alpha(k_H) \right\rangle^{m} \\ &= v_H^2 \, T(m, N, k_H) . \end{aligned} \tag{H7}$$

Here, H is taken as EGOE(k_H) with all the k_H particle matrix elements being Gaussian variables with zero center and same variance for off-diagonal matrix elements (twice for the diagonal matrix elements). This gives $\overline{(v_H^{\alpha\beta})^2} = v_H^2$ to be independent of α, β labels. It is important to note that in the dilute limit, the diagonal terms [$\alpha = \beta$ in Eq. (H7)] in the averages are neglected (as they are smaller by at least one power of $1/N$) and the individual H's are unitarily irreducible. These assumptions are no longer valid for finite-N systems and hence, evaluation of averages is more complicated. In the dilute limit, we have

$$\begin{aligned} T(m,N,k_H) &= \left\langle \sum_\alpha \alpha^\dagger(k_H) \left\{ \sum_\beta \beta(k_H)\beta^\dagger(k_H) \right\} \alpha(k_H) \right\rangle^m \\ &= \binom{\widetilde{m}+k_H}{k_H} \left\langle \sum_\alpha \alpha^\dagger(k_H)\alpha(k_H) \right\rangle^m \\ &= \binom{\widetilde{m}+k_H}{k_H}\binom{m}{k_H}. \end{aligned} \tag{H8}$$

Note that, we have used Eq. (H3) to evaluate the summation over β and Eq. (H2) to evaluate summation over α in Eq. (H8). In the 'strict' $N \to \infty$ limit, we have

$$T(m,N,k_H) \overset{N\to\infty}{\to} \binom{m}{k_H}\binom{N}{k_H}. \tag{H9}$$

In order to incorporate the finite-N corrections, we have to consider the contribution of the diagonal terms. Then, we have,

$$\begin{aligned} & T(m,N,k_H) \\ &= \left\langle \sum_{\alpha\neq\beta} \alpha^\dagger(k_H)\beta(k_H)\beta^\dagger(k_H)\alpha(k_H) \right\rangle^m + 2\left\langle \sum_\alpha \alpha^\dagger(k_H)\alpha(k_H)\alpha^\dagger(k_H)\alpha(k_H) \right\rangle^m \\ &= \left\langle \sum_\alpha \alpha^\dagger(k_H) \left\{ \sum_\beta \beta(k_H)\beta^\dagger(k_H) \right\} \alpha(k_H) \right\rangle^m \\ &+ \left\langle \sum_\alpha \alpha^\dagger(k_H)\alpha(k_H)\alpha^\dagger(k_H)\alpha(k_H) \right\rangle^m \\ &= \binom{\widetilde{m}+k_H}{k_H}\binom{m}{k_H} + \binom{m}{k_H} = \binom{m}{k_H}\left[\binom{\widetilde{m}+k_H}{k_H}+1\right]. \end{aligned} \tag{H10}$$

Note that the prefactor '2' in the second term of first line in Eq. (H10) comes because variance of the diagonal terms is twice that of the off-diagonal terms. Also, the trace $\sum_\alpha \alpha^\dagger(k_H)\alpha(k_H)\alpha^\dagger(k_H)\alpha(k_H) = \sum_\alpha \alpha^\dagger(k_H)\alpha(k_H)$ as the operator $\alpha^\dagger(k_H)\alpha(k_H)$

conserves the number of particles. Now we turn to evaluating fourth order averages.

For averages involving product of four operators of the form

$$\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m ,$$

with operators H and G independent and of body ranks k_H and k_G respectively, there are two possible ways of evaluating this trace. Either (a) first contract the H operators across the G operator using Eq. (H5) and then contract the G operators using Eq. (H4), or (b) first contract the G operators across the H operator using Eq. (H5) and then contract the H operators using Eq. (H5). Following (a), in the dilute limit, we get

$$\begin{aligned}
&\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m} \\
&= \sum_{\alpha,\beta} \overline{\left\{v_H^{\alpha\beta}\right\}^2} \left\langle \alpha^\dagger(k_H)\beta(k_H)G(k_G)\beta^\dagger(k_H)\alpha(k_H)G(k_G)\right\rangle^m \\
&= v_H^2 \binom{\widetilde{m}+k_H-k_G}{k_H} \sum_{\alpha} \left\langle \alpha^\dagger(k_H)G(k_G)\alpha(k_H)G(k_G)\right\rangle^m \\
&= v_H^2 \binom{\widetilde{m}+k_H-k_G}{k_H}\binom{m-k_G}{k_H} \langle G(k_G)G(k_G)\rangle^m \\
&= v_H^2\, v_G^2 \binom{\widetilde{m}+k_H-k_G}{k_H}\binom{m-k_G}{k_H}\binom{\widetilde{m}+k_G}{k_G}\binom{m}{k_G} .
\end{aligned} \tag{H11}$$

Similarly, following (b), in the dilute limit, we get

$$\begin{aligned}
&\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m} \\
&= v_H^2\, v_G^2 \binom{\widetilde{m}+k_G-k_H}{k_G}\binom{m-k_H}{k_G}\binom{\widetilde{m}+k_H}{k_H}\binom{m}{k_H} .
\end{aligned} \tag{H12}$$

The result should be independent of the preference. In other words, the average should have the $k_H \leftrightarrow k_G$ symmetry. As seen from Eqs. (H11) and (H12), this symmetry is violated except for the trivial case of $k_H = k_G$. However, the $k_H \leftrightarrow k_G$ symmetry is valid for 'strict' $N \to \infty$ result and also for the result incorporating finite N

corrections as discussed below. In general, the final result can be expressed as,

$$\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m} = v_H^2\, v_G^2\, F(m,N,k_H,k_G)\,. \tag{H13}$$

In the 'strict' dilute limit, both Eqs. (H11) and (H12) reduce to give result for $F(m,N,k_H,k_G)$,

$$F(m,N,k_H,k_G) = \binom{m-k_H}{k_G}\binom{m}{k_H}\binom{N}{k_H}\binom{N}{k_G}, \tag{H14}$$

In order to obtain finite-N corrections to $F(\cdots)$, we have to contract over operators whose lower symmetry parts can not be ignored. The operator $H(k_H)$ contains irreducible symmetry parts $\mathscr{F}(s)$ denoted by $s = 0,1,2,\ldots,k_H$ with respect to the unitary group $SU(N)$ decomposition of the operator. For a k_H-body number conserving operator [Ch-71, Mo-75],

$$H(k_H) = \sum_{s=0}^{k_H} \binom{m-s}{k_H-s} \mathscr{F}(s)\,.. \tag{H15}$$

Here, the $\mathscr{F}(s)$ are orthogonal with respect to m-particle averages, i.e., $\left\langle \mathscr{F}(s)\mathscr{F}^\dagger(s')\right\rangle^m = \delta_{ss'}$. Now, the m-particle trace in Eq. (H11) with binary correlations will have four parts,

$$\begin{array}{l}
\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m} \\
= v_H^2 v_G^2 \displaystyle\sum_{\alpha,\beta,\gamma,\delta} \left\langle \alpha^\dagger(k_H)\beta(k_H)\gamma^\dagger(k_G)\delta(k_G)\beta^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\gamma(k_G)\right\rangle^m \\
+ v_H^2 v_G^2 \displaystyle\sum_{\alpha,\gamma,\delta} \left\langle \alpha^\dagger(k_H)\alpha(k_H)\gamma^\dagger(k_G)\delta(k_G)\alpha^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\gamma(k_G)\right\rangle^m \\
+ v_H^2 v_G^2 \displaystyle\sum_{\alpha,\beta,\gamma} \left\langle \alpha^\dagger(k_H)\beta(k_H)\gamma^\dagger(k_G)\gamma(k_H)\beta^\dagger(k_H)\alpha(k_H)\gamma^\dagger(k_G)\gamma(k_G)\right\rangle^m \\
+ v_H^2 v_G^2 \displaystyle\sum_{\alpha,\delta} \left\langle \alpha^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\delta(k_G)\alpha^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\delta(k_G)\right\rangle^m \\
= X + Y_1 + Y_2 + Z\,.
\end{array} \tag{H16}$$

Note that we have decomposed each operator into diagonal and off-diagonal parts. We have used the condition that the variance of the diagonal matrix elements is twice that of the off-diagonal matrix elements in the defining spaces to convert the restricted summations into unrestricted summations appropriately to obtain the four terms in the RHS of Eq. (H17). Following Mon's thesis [Mo-73] and applying unitary decomposition to $\gamma\delta^\dagger$ (also $\delta\gamma^\dagger$) in the first two terms and $\alpha\beta^\dagger$ (also $\beta\alpha^\dagger$) in the third term we get X, Y_1 and Y_2. To make things clear, we will discuss the derivation for X term in detail before proceeding further. Applying unitary decomposition to the operators $\gamma^\dagger(k_G)\delta(k_G)$ and $\gamma(k_G)\delta^\dagger(k_G)$ using Eq. (H15), we have

$$X = v_H^2\, v_G^2 \sum_{\alpha,\beta,\gamma,\delta} \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \left\langle \alpha^\dagger(k_H)\beta(k_H)\mathscr{F}^\dagger_{\gamma\delta}(s)\beta^\dagger(k_H)\alpha(k_H)\mathscr{F}_{\gamma\delta}(s)\right\rangle^m . \quad \text{(H17)}$$

Contracting the operators $\beta\beta^\dagger$ across $\mathscr{F}$'s using Eq. (H5) and operators $\alpha^\dagger\alpha$ across $\mathscr{F}$ using Eq. (H4) gives,

$$X = v_H^2\, v_G^2 \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \binom{\tilde{m}+k_H-s}{k_H} \binom{m-s}{k_H} \sum_{\gamma,\delta} \left\langle \mathscr{F}^\dagger_{\gamma\delta}(s)\mathscr{F}_{\gamma\delta}(s)\right\rangle^m . \quad \text{(H18)}$$

Inversion of the equation,

$$\sum_{\gamma,\delta} \left\langle \gamma^\dagger(k_G)\delta(k_G)\delta^\dagger(k_G)\gamma(k_G)\right\rangle^m = Q(m) = \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \sum_{\gamma,\delta} \left\langle \mathscr{F}^\dagger_{\gamma\delta}(s)\mathscr{F}_{\gamma\delta}(s)\right\rangle^m , \quad \text{(H19)}$$

gives,

$$\begin{aligned} &\binom{m-s}{k_G-s}^2 \sum_{\gamma,\delta} \left\langle \mathscr{F}^\dagger_{\gamma\delta}(s)\mathscr{F}_{\gamma\delta}(s)\right\rangle^m \\ &= \binom{m-s}{k_G-s}^2 \binom{N-m}{s} \binom{m}{s} [(k_G-s)!s!]^2 (N-2s+1) \qquad \text{(H20)} \\ &\times \sum_{t=0}^{s} \frac{(-1)^{t-s}[(N-t-k_G)!]^2}{(s-t)!(N-s-t+1)!t!(N-t)!} Q(N-t) . \end{aligned}$$

It is important to mention that there are errors in the equation given in Mon's thesis and we have verified Eq. (H20) using Mathematica (Mon = Eq. (H20) / $[(N-2s)!(s!)^2]$).

For the average required in Eq. (H19), we have

$$Q(m)=\sum_{\gamma,\delta}\left\langle \gamma^{\dagger}(k_G)\delta(k_G)\delta^{\dagger}(k_G)\gamma(k_G)\right\rangle^{m}=\binom{\tilde{m}+k_G}{k_G}\binom{m}{k_G}. \tag{H21}$$

Simplifying Eq. (H20) using Eq. (H21) and using the result in Eq. (H18) along with the series summation

$$\sum_{t=0}^{s}\frac{(-1)^{t-s}(N-t-k_G)!\,(k_G+t)!}{(s-t)!\,(t!)^2\,(N-s-t+1)!}=\frac{k_G!(N-k_G-s)!}{(N+1-s)!}\binom{k_G}{s}\binom{N+1}{s}, \tag{H22}$$

the expression for X is,

$$\begin{aligned} X &= v_H^2 v_G^2\, F(m,N,k_H,k_G)\,; \\ F(m,N,k_H,k_G) &= \sum_{s=0}^{k_G}\binom{m-s}{k_G-s}^2\binom{\tilde{m}+k_H-s}{k_H}\binom{m-s}{k_H}\binom{\tilde{m}}{s}\binom{m}{s}\binom{N+1}{s} \\ &\times \frac{N-2s+1}{N-s+1}\binom{N-s}{k_G}^{-1}\binom{k_G}{s}^{-1}. \end{aligned} \tag{H23}$$

Although not obvious, X has $k_H \leftrightarrow k_G$ symmetry and we have verified this explicitly for $k_H, k_G \leq 2$. Similarly, the terms Y_1 and Y_2 are given by,

$$\begin{aligned} Y_1 &= v_H^2 v_G^2\, B(m,N,k_H,k_G)\,,\quad Y_2 = v_H^2 v_G^2\, B(m,N,k_G,k_H)\,; \\ B(m,N,k_H,k_G) &= \sum_{s=0}^{k_G}\binom{m-s}{k_G-s}^2\binom{\tilde{m}+k_H-s}{k_H}\binom{m-s}{k_H}\binom{\tilde{m}}{s}\binom{m}{s} \\ &\times \frac{N-2s+1}{N-s+1}\binom{N-s}{k_G}^{-1}\binom{k_G}{s}^{-1}. \end{aligned} \tag{H24}$$

In order to derive Eq.(H24), we have used $Q(m)=\binom{m}{k_G}$ along with the series summation,

$$\sum_{t=0}^{s}\frac{(-1)^{t-s}(N-t-k_G)!\,k_G!\,t!}{(s-t)!\,(t!)^2\,(N-s-t+1)!}=\frac{k_G!(N-k_G-s)!}{(N+1-s)!}\binom{k_G}{s}. \tag{H25}$$

Note that Mon's thesis gives $\binom{m-s}{s}$ in place of $\binom{m-s}{k}$ with $k=k_H$ or k_G for X, Y_1 and Y_2

in Eqs. (H23) and (H24) and it should be a printing error. The expressions given in Eqs. (H23) and (H24) agree with the results given in Tomsovic's thesis [To-86]. Finally, the result for Z is

$$
\begin{aligned}
Z &= v_H^2 v_G^2 \sum_{\alpha,\delta} \left\langle \alpha^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\delta(k_G)\alpha^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\delta(k_G)\right\rangle^m \\
&= v_H^2 v_G^2 \sum_{\alpha} \left\langle \alpha^\dagger(k_H)\alpha(k_H)\right\rangle^m \sum_{\delta} \left\langle \delta^\dagger(k_G)\delta(k_G)\right\rangle^m \\
&= v_H^2 v_G^2\, C(m,N,k_H,k_G)\,; \\
C(m,N,k_H,k_G) &= \binom{m}{k_H}\binom{m}{k_G}\,.
\end{aligned} \tag{H26}
$$

Equation (H26) is in agreement with the result in Mon's thesis with $k_H = k_G = k$. However, it differs from the result given in Tomsovic's thesis. For a one-body operator, obviously $Z = m^2$ and this confirms that Eq. (H26) is correct. Therefore Eqs. (H16)-(H26) give the final formula for the trace $\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m}$. It is easily seen that dominant contribution to the average $\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m}$ comes from the X term and therefore, in all the applications, we use

$$
\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m} = X = v_H^2\, v_G^2\, F(m,N,k_H,k_G)\,. \tag{H27}
$$

An immediate application of these averages is in evaluating the fourth order average $\overline{\left\langle H^4(k_H)\right\rangle^m}$. There will be three different correlation patterns that will contribute to this average in the binary correlation approximation (we must correlate in pairs the operators for all moments of order > 2),

$$
\begin{aligned}
\overline{\left\langle H^4(k_H)\right\rangle^m} &= \overline{\langle H(k_H)H(k_H)H(k_H)H(k_H)\rangle^m} \\
&+ \overline{\langle H(k_H)H(k_H)H(k_H)H(k_H)\rangle^m} \\
&+ \overline{\langle H(k_H)H(k_H)H(k_H)H(k_H)\rangle^m}\,.
\end{aligned} \tag{H28}
$$

In Eq. (H28), we denote the correlated pairs of operators by same color in each pattern. The first two terms on the RHS of Eq. (H28) are equal due to cyclic invariance and follow easily from Eq. (H7),

$$\begin{aligned}
\overline{\langle H(k_H)H(k_H)H(k_H)H(k_H)\rangle^m} &= \overline{\langle H(k_H)H(k_H)H(k_H)H(k_H)\rangle^m} \\
&= \left[\overline{\left\langle H^2(k_H)\right\rangle^m}\right]^2 .
\end{aligned} \tag{H29}$$

Similarly, the third term on the RHS of Eq. (H28) follows easily from Eq. (H27),

$$\overline{\langle H(k_H)H(k_H)H(k_H)H(k_H)\rangle^m} = v_H^4\, F(m,N,k_H,k_H)\,. \tag{H30}$$

Finally, $\overline{\left\langle H^4(k_H)\right\rangle^m}$ is given by,

$$\overline{\left\langle H^4(k_H)\right\rangle^m} = v_H^4\,\left[2\,\{T(m,N,k_H)\}^2 + F(m,N,k_H,k_H)\right]\,. \tag{H31}$$

Simplifying $T(\cdots)$ and $F(\cdots)$ in 'strict' $N \to \infty$ limit and using Eqs. (H7) and (H31), the excess parameter for spinless EGOE(k_H) is,

$$\gamma_2(m) = \frac{\overline{\left\langle H^4(k_H)\right\rangle^m}}{\left[\overline{\left\langle H^2(k_H)\right\rangle^m}\right]^2} - 3 = \frac{\binom{m-k_H}{k_H}}{\binom{m}{k_H}} - 1 \stackrel{m>>k_H}{\longrightarrow} -\frac{k_H^2}{m}\,. \tag{H32}$$

Equation (H32) was first derived in [Mo-75]. As seen from Eq. (H32), state densities for spinless EGOE(k_H) approach Gaussian form for large m and they exhibit, as m increases from k_H, semicircle to Gaussian transition with $m = 2k_H$ being the transition point. The results for $\overline{\left\langle H^2(k_H)\right\rangle^m}$ and $\overline{\left\langle H^4(k_H)\right\rangle^m}$ easily extend, though not obvious, to averages over two-orbit spaces with operator H having fixed body ranks in the two spaces. It is useful to mention that the details for the two-orbit averages using binary correlation approximation are not available in literature. Now, we will discuss the two-orbit results.

In many nuclear structure applications and also for applications to interacting spin systems, fourth order traces over two orbit configurations are needed. Let us consider m particles in two orbits with number of sp states being N_1 and N_2 respectively. Now the m-particle space can be divided into configurations (m_1, m_2) with m_1 particles in the #1 orbit and m_2 particles in the #2 orbit such that $m = m_1 + m_2$. Considering the operator H with fixed body ranks in m_1 and m_2 spaces such that (m_1, m_2) are preserved by this operators, the general form for H is,

$$H(k_H) = \sum_{i+j=k_H;\alpha,\beta,\gamma,\delta} \left[v_H^{\alpha\beta\gamma\delta}(i,j)\right] \alpha_1^\dagger(i)\beta_1(i)\gamma_2^\dagger(j)\delta_2(j)\,. \tag{H33}$$

Now, it is easily seen that, in the dilute limit,

$$\begin{aligned}
&\overline{\left\langle H^2(k_H)\right\rangle^{m_1,m_2}} \\
&= \sum_{i+j=k_H} v_H^2(i,j) \sum_{\alpha,\beta,\gamma,\delta} \left\langle \alpha_1^\dagger(i)\beta_1(i)\gamma_2^\dagger(j)\delta_2(j)\beta_1^\dagger(i)\alpha_1(i)\delta_2^\dagger(j)\gamma_2(j)\right\rangle^{m_1,m_2} \\
&= \sum_{i+j=k_H} v_H^2(i,j) \sum_{\alpha,\beta}\left\langle \alpha_1^\dagger(i)\beta_1(i)\beta_1^\dagger(i)\alpha_1(i)\right\rangle^{m_1} \sum_{\gamma,\delta}\left\langle \gamma_2^\dagger(j)\delta_2(j)\delta_2^\dagger(j)\gamma_2(j)\right\rangle^{m_2} \\
&= \sum_{i+j=k_H} v_H^2(i,j)\; T(m_1,N_1,i)\; T(m_2,N_2,j)\,.
\end{aligned} \tag{H34}$$

Note that $v_H^2(i,j) = \overline{[v_H^{\alpha\beta\gamma\delta}(i,j)]^2}$ and T's are defined by Eqs. (H8) and (H9). The ensemble is defined such that $v_H^{\alpha\beta\gamma\delta}(i,j)$ are independent Gaussian random variables with zero center and the variances depend only on the indices i and j. The formula for $\overline{\langle H(k_H)H(k_H)\rangle^{m_1,m_2}}$ with finite (N_1, N_2) corrections is,

$$\overline{\langle H(k_H)H(k_H)\rangle^{m_1,m_2}} = \sum_{i+j=k_H} v_H^2(i,j) \binom{m_1}{i}\binom{m_2}{j}\left[\binom{\tilde{m}_1+i}{i}\binom{\tilde{m}_2+j}{j}+1\right]. \tag{H35}$$

Similarly, with two operators H and G (with body ranks k_H and k_G respectively) that are independent and both preserving (m_1, m_2), $\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^{m_1,m_2}}$

is given by,

$$\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^{m_1,m_2}} = \sum_{i+j=k_H,\, t+u=k_G} v_H^2(i,j)\; v_G^2(t,u)\, F(m_1,N_1,i,t)\, F(m_2,N_2,j,u)\,. \tag{H36}$$

Also, extending the single orbit results with finite-N corrections, we have,

$$\begin{aligned}
&\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^{m_1,m_2}} \\
&= \sum_{i+j=k_H,\, t+u=k_G}\; \sum_{\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2,\delta_2} v_H^2(i,j)\;\; v_G^2(t,u) \\
&\times \left\langle \alpha_1^\dagger(i)\beta_1(i)\gamma_1^\dagger(t)\delta_1(t)\beta_1^\dagger(i)\alpha_1(i)\delta_1^\dagger(t)\gamma_1(t)\right\rangle^{m_1} \\
&\times \left\langle \alpha_2^\dagger(j)\beta_2(j)\gamma_2^\dagger(u)\delta_2(u)\beta_2^\dagger(j)\alpha_2(j)\delta_2^\dagger(u)\gamma_2(u)\right\rangle^{m_2}\,.
\end{aligned} \tag{H37}$$

Applying Eqs. (H16)-(H26) to the two traces in Eq. (H37), we get

$$\begin{aligned}
\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^{m_1,m_2}} &= \sum_{i+j=k_H,\, t+u=k_G} v_H^2(i,j)\; v_G^2(t,u) \\
\times \big[F(m_1,N_1,i,t)F(m_2,N_2,j,u) &+ B(m_1,N_1,i,t)B(m_2,N_2,j,u) \\
+ B(m_1,N_1,t,i)B(m_2,N_2,u,j) &+ C(m_1,N_1,i,t)C(m_2,N_2,j,u)\big]\,.
\end{aligned} \tag{H38}$$

The $F(\cdots)$'s appearing in Eq. (H38) are given by Eq. (H23). Also, the B's and C's are given by Eqs. (H24) and (H26) respectively. Finally, in the strict dilute limit as $F(\cdots)$'s dominate over B's and C's, we get back Eq. (H36). In all the applications discussed in Chapter 7, we use Eq. (H36). Now, using Eqs. (H34) and (H36), we have

$$\begin{aligned}
\overline{\left\langle H^4(k_H)\right\rangle^{m_1,m_2}} &= 2\left[\sum_{i+j=k_H} v_H^2(i,j)\; T(m_1,N_1,i)\; T(m_2,N_2,j)\right]^2 \\
&+ \sum_{i+j=k_H,\, t+u=k_H} v_H^2(i,j)\; v_H^2(t,u)\, F(m_1,N_1,i,t)\, F(m_2,N_2,j,u)\,.
\end{aligned} \tag{H39}$$

As a simple application of Eqs. (H34) and (H39), let us consider $\gamma_2(m, M_S)$ for EGOE(2)-M_S ensemble. For this ensembles, H will preserve M_S and it is defined for a system of m fermions carrying spin $\mathbf{s} = \frac{1}{2}$ degree of freedom (see also Appendix G). Then, we have two orbits with $N_1 = N_2 = \Omega$, $m_1 = m/2 + M_S$ and $m_2 = m/2 - M_S$. Here, orbit #1 corresponds to sp states with $m_\mathbf{s} = +\frac{1}{2}$ and orbit #2 corresponds to sp states with $m_\mathbf{s} = -\frac{1}{2}$. Note that the fixed-$M_S$ dimension is $D(m, M_S) = \binom{\Omega}{m/2-M_S}\binom{\Omega}{m/2+M_S}$. By substituting $m_1 = m/2 + M_S$ and $m_2 = m/2 - M_S$, Eqs. (H34) and (H39) will give $\overline{\langle H^4(2)\rangle^{m,M_S}}$ and $\overline{\langle H^2(2)\rangle^{m,M_S}}$, respectively. Then, the fixed-(m, M_S) excess parameter $\gamma_2(m, M_S)$ in the dilute limit is given by,

$$\gamma_2(m, M_S) = \frac{\displaystyle\sum_{i+j=2,\, t+u=2} v_H^2(i,j)\; v_H^2(t,u)\; F(m_1, \Omega, i, t)\; F(m_2, \Omega, j, u)}{\left[\displaystyle\sum_{i+j=2} v_H^2(i,j)\; T(m_1, \Omega, i)\; T(m_2, \Omega, j)\right]^2} - 1\,, \tag{H40}$$

with $T(\cdots)$'s and $F(\cdots)$'s given before.