Chapter 1

Introduction

1.1 Random Matrix Theory, Quantum Chaos and Finite Quantum Systems

Random matrix theory (RMT), starting with Wigner and Dyson's Gaussian random ensembles introduced to describe neutron resonance data, has emerged as a powerful statistical approach leading to paradigmatic models describing generic properties of complex systems. Importance of RMT has been recognized almost since its inception in physics by Wigner in 1955 [Wi-55] to explain the compound nucleus resonance data. Though random matrices were encountered much earlier in 1928 by Wishart [Wi-28] in the context of multivariate statistics and later in 1967 by Pastur [Ma-67], their extensive study began with the pioneering work of Wigner [Wi-67, Po-65]. Though not referred explicitly, Bohr's idea of compound nucleus [Bo-36] almost certainly motivated Wigner to introduce random matrices. Porter's book [Po-65] gives a good introduction to classical random matrix ensembles with a detailed derivation of the joint probability distribution for these ensembles along with an impressive and instructive collection of papers on the subject till 1965. Mathematical foundations of RMT are well described by Mehta [Me-04] (the first edition of Mehta's book was published in 1967). The classical random matrix ensembles are developed and applied during 1955-1972 by Dyson, Mehta, Porter and others [Po-65, Br-81]. In the last three decades, RMT has been successfully used in diverse areas, as shown in Fig. 1.1, with wide ranging applicability to various mathematical, physical and en-

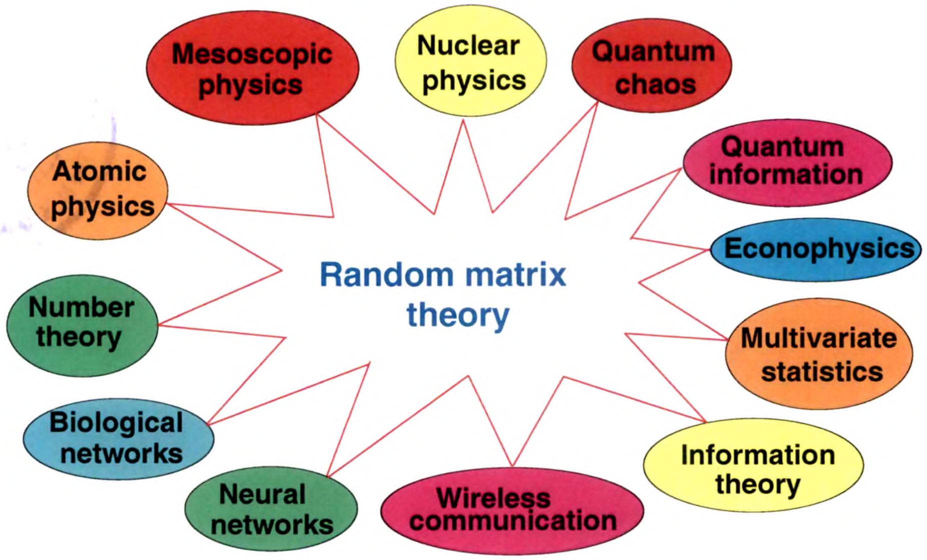

Figure 1.1: Figure showing wide ranging applications of random matrix theory.

gineering branches [Po-65, Me-04, Br-81, Gu-98, Tu-04, St-06, Br-06, Ul-08, We-09, Fo-10, Wr-10, An-10, Ba-10, Ha-10, Mi-10].

RMT helps to analyze statistical properties of physical systems whose exact Hamiltonian is too complex to be studied directly. The exact Hamiltonian of the system under consideration is represented by an ensemble of random matrices that incorporate generic symmetry properties of the system. As stated by Wigner [Wi-61]: *The assumption is that the Hamiltonian which governs the behavior of a complicated system is a random symmetric matrix, with no special properties except for its symmetric nature.* More importantly, as emphasized by French [Fr-95]: *with one short step beyond this, specifically replacing "complicated" by "non-integrable", this paper would have led to the foundations of quantum chaos. Perhaps it should be so regarded even as it stands.* Depending on the global symmetry properties, namely rotational and time-reversal, Dyson classified the classical random matrix ensembles into three classes-Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles [Dy-62]. As the names suggest, these ensembles will be invariant under orthogonal, unitary, and symplectic transformations, respectively. The corresponding matrices will be real symmetric, complex hermitian and real quaternion matrices. In order to study symmetry breaking effects on level and strength fluctuations and order-chaos transitions, it is necessary to consider interpolating or deformed random matrix ensembles [Pa-81, Fr-88]. Earliest examples include banded random matrix ensembles, the Porter-Rosenzweig model and 2×2 GOE due to Dyson [Po-65].

RMT has been established to be one of the central themes in quantum physics with the recognition that quantum systems whose classical analogues are chaotic, follow RMT. The BGS conjecture [Bo-84a] is the corner stone for this and the earlier work on this is due to McDonald and Kaufman [Mc-79], Casati et al [Ca-80] and Berry [Be-81]. The BGS conjecture is: *Spectra of time-reversal-invariant systems whose classical analogues are K systems show the same fluctuation properties as predicted by GOE.* Also as stated by BGS: *if this conjecture happens to be true, it will then have established the 'universality of the laws of level fluctuations' in quantal spectra already found in nuclei and to a lesser extent in atoms. Then, they should also be found in other quantal systems, such as molecules, hadrons etc.* The details of the developments establishing the connection between RMT and the spectral fluctuation properties of quantum systems whose classical analogues are chaotic are summarized in [Ha-10, St-06]. Recently, Haake et al gave a proof for the BGS conjecture using semi-classical methods [He-07]. Combining BGS work with that of Berry on integrable systems [Be-77], as summarized by Altshuler in the abstract of the colloquium he gave in memory of J.B. French at the university of Rochester in 2004: *Classical dynamical systems can be separated into two classes - integrable and chaotic. For quantum systems this distinction manifests itself, e.g. in spectral statistics. Roughly speaking integrability leads to Poisson distribution for the energies while chaos implies Wigner-Dyson statistics of levels, which are characteristic for the ensemble of random matrices. The onset of chaotic behavior for a rather broad class of systems can be understood as a delocalization in the space of quantum numbers that characterize the original integrable system ...* Following this, as stated by Papenbrock and Weidenmüller [Pa-07]: *We speak of chaos in quantum systems if the statistical properties of the eigenvalue spectrum coincide with predictions of random-matrix theory.* For example, the nearest neighbor spacing distribution (NNSD) showing von-Neumann Wigner [Ne-29] 'level repulsion' and the Dyson-Mehta [Dy-63] Δ_3 statistic showing 'spectral rigidity' are exhibited by quantum systems whose classical analogues are chaotic; see Fig. 1.2. It is now well recognized that chaos is a typical feature of atomic nuclei and other self bound Fermi systems.

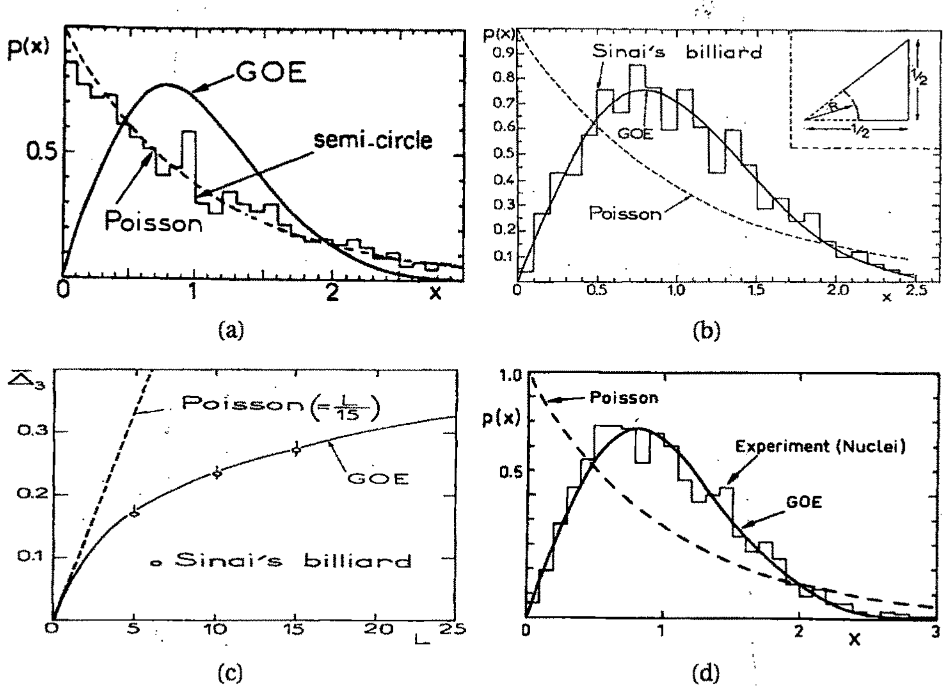

Figure 1.2: Figure illustrating the connection between RMT and quantum chaos. (a) NNSD for the regular (integrable) semi-circular billiard follows Poisson distribution $P(X)dX = e^{-X}dX$ (taken from [Bo-84]); (b) NNSD for the chaotic Sinai billiard follows GOE (taken from [Bo-84a]) and the GOE Wigner form is $P(X)dX = \frac{\pi}{2}X\exp-\frac{\pi X^2}{4}dX$; (c) Dyson-Mehta statistic $\Delta_3(L)$ for Sinai billiard also follows GOE (taken from [Bo-84a]) and for GOE, $\Delta_3(L) \sim \ln L$; and (d) NNSD for the nuclear data ensemble (NDE) follows GOE [Bo-83]. Though we haven't shown, the $\Delta_3(L)$ for $L \leq 20$ also follows GOE for the NDE [Ha-82a]. Note that X is the level spacing normalized to the average level spacing and L is the length of the energy interval over which Δ_3 is calculated. It is clearly seen that, unlike regular billiard, chaotic billiard follows RMT and more importantly, the NDE follows RMT establishing that the neutron resonance region is a region of chaos.

Finite quantum systems such as nuclei, atoms, quantum dots, small metallic grains, interacting spin systems modeling quantum computing core and BEC, share one common property - their constituents (predominantly) interact via two-particle interactions. As pointed out by French [Fr-80]: *The GOE, now almost universally regarded as a model for a corresponding chaotic system is an ensemble of multi-body, not two-body interactions. This difference shows up both in one-point (density of states) and two-point (fluctuations) functions generated by nuclear shell model.* Therefore, it is more appropriate to represent an isolated finite interacting quantum system by random matrix models generated by random *two-body* interactions (in general, by k-body interactions with $k <$ particle number m). Matrix ensembles generated by random two-body interactions, called two-body random ensembles (TBRE), model what one may call many-body chaos or stochasticity or complexity exhibited by these systems. These ensembles are defined by representing the two-particle Hamiltonian by one of the classical ensembles (GOE or GUE or GSE) and then the $m > 2$ particle H matrix is generated by the m-particle Hilbert space geometry [Fr-70, Bo-71, Mo-75] and with GOE(GUE) embedding, they will be EGOE(EGUE). The key element here is the recognition that there is a Lie algebra that transports the information in the two-particle spaces to many-particle spaces [Mo-75, Ko-05, Be-01a]. Thus the random matrix ensemble in the two-particle spaces is embedded in the m-particle H matrix and therefore these ensembles are more generically called embedded ensembles (EE) [Mo-75, Br-81]. Due to the two-body selection rules, many of the m-particle matrix elements will be zero. Figure 1.3 gives an example of a H-matrix displaying the structure due to two-body selection rules which form the basis for the EE description. At this stage, it is appropriate to recall the purpose, as stated by the organizers Altshuler, Bohigas and Weidenmüller, of a workshop (held at ECT*, Trento in February 1997) on chaotic dynamics of many-body systems: *The study of quantum manifestations of classical chaos has known important developments, particularly for systems with few degrees of freedom. Now, we understand much better how the universal and system-specific properties of 'simple chaotic systems' are connected with the underlying classical dynamics. The time has come to extend, from this perspective, our understanding to objects with many degrees of freedom, such as interacting many-body systems. Problems of nuclear, atomic, and molecular theory as well as the theory of mesoscopic systems will be discussed at the workshop.* Note that, chaos implies RMT and the new emphasis is on many-body chaos. Recent thinking is that EE generate paradigmatic models for many-body chaos [Ko-01, Go-11] (one-body chaos is well understood using classical ensembles). The present thesis is devoted to developing and analyzing a variety of EE so that one can quantify and apply the results of many-body chaos.

Simplest of EE is the embedded Gaussian orthogonal ensemble of random matrices for spinless fermion/boson systems generated by random two-body interactions. However, unlike for fermion systems, there are only a few EE investigations for finite interacting boson systems [Pa-00, Ag-01, Ag-02, Ch-03, Ch-04]; the corresponding EE are called BEE (B stands for bosons). The spinless fermion/boson EGEs (orthogonal

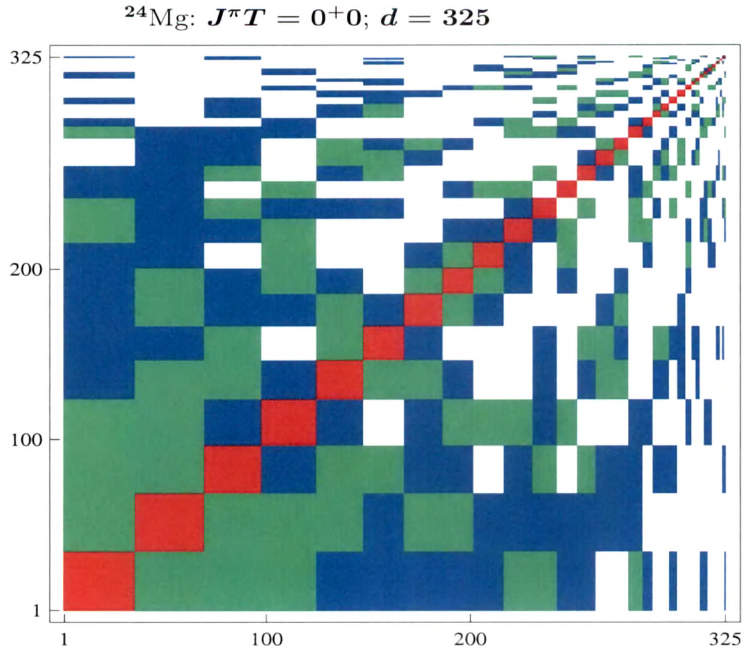

Figure 1.3: Block matrix structure of the H matrix of ^{24}Mg displaying two-body selection rules. Total number of blocks are 33, each labeled by the spherical configurations (m_1, m_2, m_3). The diagonal blocks are shown in red and within these blocks there will be no change in the occupancy of the nucleons in the three sd orbits. Green corresponds to the region (in the matrix) connected by the two-body interaction that involve change of occupancy of one nucleon. Similarly, blue corresponds to change of occupancy of two nucleons. Finally, white correspond to the region forbidden by the two-body selection rules. This figure was first reported by us in [Ma-10c] and a similar figure was given earlier in [Pa-05] for ^{28}Si with $(J^{\pi}T) = (0^{+}0)$. Section 8.2 gives further discussion.

and unitary versions) have been explored in detail from 70's [Br-81,Ko-01,Go-11] with a major revival from mid 90's [Fl-94, Fl-96, Fl-97, Ho-95, Fr-96, Ja-97, Ko-98]. Before proceeding further, we briefly describe the known results for spinless fermion/boson EGEs for completeness and for easy reference in the following chapters.

1.2 Embedded Ensembles for Spinless Fermion Systems

The embedding algebra for EGOE(k) and EGUE(k) [also BEGOE(k) and BEGUE(k)] for a system of m spinless particles (fermions or bosons) in N single particle (sp) states with k-body interactions ($k < m$) is $SU(N)$. These ensembles are defined by the three parameters (N, m, k). A large number of asymptotic results are derived for EGOE(k) and EGUE(k) using Wigner's binary correlation approximation [Mo-75, Br-81, Fr-88] and, more importantly, also some exact analytical results are derived using

$SU(N)$ Wigner-Racah algebra [Ko-05, Be-01a, Pl-02]. For bosons, the dense limit studies are interesting as this limit does not exist for fermion systems [Ko-80, Pa-00, Ag-01, Ag-02, Ch-03, Ch-04]. Now we will briefly discuss the definition, construction and the known results for these ensembles.

1.2.1 EGOE(2) and EGOE(k) ensembles

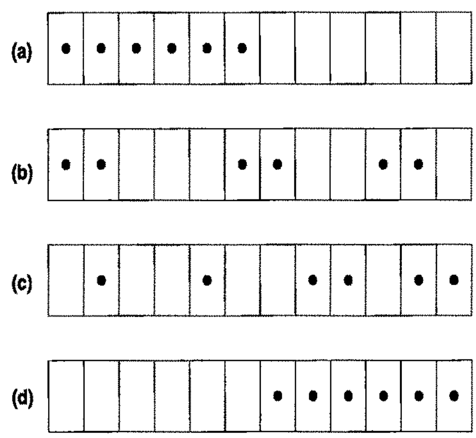

Figure 1.4: Figure showing some configurations for the distribution of $m = 6$ spinless fermions in $N = 12$ single particle states. Distributing the m fermions in all possible ways in N single particle states generates the m-particle configurations or basis states. This is similar to distributing m particles in N boxes with the conditions that the occupancy of each box can be either zero or one and the total number of occupied boxes equals m. In the figure, (a) corresponds to the basis state $|\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\rangle$, (b) corresponds to the basis state $|\nu_1\nu_2\nu_6\nu_7\nu_{10}\nu_{11}\rangle$, (c) corresponds to the basis state $|\nu_2\nu_5\nu_8\nu_9\nu_{11}\nu_{12}\rangle$ and (d) corresponds to the basis state $|\nu_7\nu_8\nu_9\nu_{10}\nu_{11}\nu_{12}\rangle$.

The EGOE(2) ensemble for spinless fermion systems is generated by defining the two-body Hamiltonian H to be GOE in two-particle spaces and then propagating it to many-particle spaces by using the geometry of the many-particle spaces [this is in general valid for k-body Hamiltonians, with $k < m$, generating EGOE(k)]. Let us consider a system of m spinless fermions occupying N sp states. Each possible distribution of fermions in the sp states generates a configuration or a basis state; see Fig. 1.4. Given the sp states $|\nu_i\rangle$, $i = 1, 2, \ldots, N$, the EGOE(2) is defined by the Hamiltonian operator,

$$\widehat{H} = \sum_{\nu_i<\nu_j,\, \nu_k<\nu_\ell} \langle \nu_k\, \nu_\ell \mid \widehat{H} \mid \nu_i\, \nu_j \rangle\, a^\dagger_{\nu_\ell}\, a^\dagger_{\nu_k}\, a_{\nu_i}\, a_{\nu_j}\,. \tag{1.2.1}$$

The action of the Hamiltonian operator defined by Eq. (1.2.1) on the basis states

$|\nu_1\nu_2\cdots\nu_m\rangle$ (see Fig. 1.4 for examples) generates the EGOE(2) ensemble in m-particle spaces. The symmetries for the antisymmetrized two-body matrix elements $\langle \nu_k\, \nu_\ell \mid \widehat{H} \mid \nu_i\, \nu_j \rangle$ are

$$\begin{aligned} \langle \nu_k\, \nu_\ell \mid \widehat{H} \mid \nu_j\, \nu_i \rangle &= -\langle \nu_k\, \nu_\ell \mid \widehat{H} \mid \nu_i\, \nu_j \rangle\,, \\ \langle \nu_k\, \nu_\ell \mid \widehat{H} \mid \nu_i\, \nu_j \rangle &= \langle \nu_i\, \nu_j \mid \widehat{H} \mid \nu_k\, \nu_\ell \rangle\,. \end{aligned} \tag{1.2.2}$$

Note that a_{ν_i} and $a_{\nu_i}^\dagger$ in Eq. (1.2.1) annihilate and create a fermion in the sp state $|\nu_i\rangle$, respectively. The Hamiltonian matrix $H(m)$ in m-particle spaces contains three different types of non-zero matrix elements (all other matrix elements are zero due to two-body selection rules) and explicit formulas for these are [Ko-01],

$$\begin{aligned} \langle \nu_1\nu_2\cdots\nu_m \mid \widehat{H} \mid \nu_1\nu_2\cdots\nu_m \rangle &= \sum_{\nu_i<\nu_j\leq\nu_m} \langle \nu_i\nu_j \mid \widehat{H} \mid \nu_i\nu_j \rangle\,, \\ \langle \nu_p\nu_2\nu_3\cdots\nu_m \mid \widehat{H} \mid \nu_1\nu_2\cdots\nu_m \rangle &= \sum_{\nu_i=\nu_2}^{\nu_m} \langle \nu_p\nu_i \mid \widehat{H} \mid \nu_1\nu_i \rangle\,, \\ \langle \nu_p\nu_q\nu_3\cdots\nu_m \mid \widehat{H} \mid \nu_1\nu_2\nu_3\cdots\nu_m \rangle &= \langle \nu_p\nu_q \mid \widehat{H} \mid \nu_1\nu_2 \rangle\,. \end{aligned} \tag{1.2.3}$$

Note that, in Eq. (1.2.3), the notation $|\nu_1\nu_2\cdots\nu_m\rangle$ denotes the orbits occupied by the m spinless fermions. The EGOE(2) is defined by Eqs. (1.2.2) and (1.2.3) with GOE representation for $\widehat{H}$ in the two-particle spaces, i.e.,

$$\begin{aligned} &\langle \nu_k\, \nu_\ell \mid \widehat{H} \mid \nu_i\, \nu_j \rangle \;\text{ are independent Gaussian random variables} \\ &\overline{\langle \nu_k\, \nu_\ell \mid \widehat{H} \mid \nu_i\, \nu_j \rangle} = 0\,, \\ &\overline{|\langle \nu_k\, \nu_\ell \mid \widehat{H} \mid \nu_i\, \nu_j \rangle|^2} = v^2\left(1+\delta_{(ij),(k\ell)}\right). \end{aligned} \tag{1.2.4}$$

In Eq. (1.2.4), 'overline' indicates ensemble average and v is a constant. Now the m-fermion EGOE(2) Hamiltonian matrix ensemble is denoted by $\{H(m)\}$ where $\{\ldots\}$ denotes ensemble, with $\{H(2)\}$ being GOE. Note that, the m-particle H-matrix dimension is $d_f(N,m) = \binom{N}{m}$ and the number of independent matrix elements is $d_f(N,2)[d_f(N,2)+1]/2$; the subscript 'f' in $d_f(N,m)$ stands for 'fermions'. A computer code for constructing EGOE(2) ensemble is available in our group [Ko-01]; many other groups in the world have also developed codes for EGOE(2).

Just as the EGOE(2) ensemble, it is possible to define k-body ($k < m$) ensembles EGOE(k) (these are also called 2-BRE, 3-BRE, $\cdots$ in [Vo-08]). Some of the generic results, derived numerically and analytically, for EGOE(k) are as follows: (i) state densities approach Gaussian form for large m and they exhibit, as m increases from k, semi-circle to Gaussian transition with $m = 2k$ being the transition point [Br-81, Be-01a]; (ii) level and strength fluctuations follow GOE (as far as one can infer from numerical examples) [Br-81]; (iii) there is average fluctuation separation with increasing m and the averages are determined by a few long wavelength modes in the normal mode decomposition of the density of states [Mo-75, Br-81, Le-08]; (iv) smoothed (ensemble averaged) transition strength densities take bivariate Gaussian form and as a consequence transition strength sums originating from a given eigenstate will be close to a ratio of two Gaussians [Fr-88]; (v) cross-correlations between spectra with different particle numbers will be non-zero [Pa-06, Ko-06a]. For reviews on EGOE, see [Br-81, Be-03, Ko-01].

1.2.2 EGOE(1+2) ensemble

Besides the two-body interaction, Hamiltonians for realistic systems also contain a mean field one-body part (generating shell structure) and therefore a more appropriate random matrix ensemble for finite quantum systems is EGOE(1+2)[a], the embedded GOE of one plus two-body interactions [Fl-96, Fl-96a, Fl-97, Ko-01]. Given the mean-field Hamiltonian $\widehat{h}(1) = \sum_i \epsilon_i \hat{n}_i$, where $\hat{n}_i$ are number operators and ϵ_i, $i = 1, 2, \ldots, N$ are the sp energies, and the two-body interaction $\widehat{V}(2)$ (this is same as $\widehat{H}(2)$ defined in Sec. 1.2.1), EGOE(1+2) is defined by the operator

$$\{\widehat{H}\} = \widehat{h}(1) + \lambda \{\widehat{V}(2)\} . \tag{1.2.5}$$

[a]At this point it is also useful to mention that EGOE(1+2)'s [and EGOE(2)'s] are also called TBRE in literature; Sec. 5.7 in [Go-11] gives clarifications on this nomenclature. As Brody et al state [Br-81]: *The most severe mathematical difficulties with TBRE are due to angular momentum constraints ... Another type of ensemble, ... much closer to being mathematical tractable abandons the J restrictions entirely ... an embedded GOE, or EGOE for short.*

The $\{\widehat{V}(2)\}$ ensemble in two-particle spaces is represented by GOE(1) and λ is the strength of $\widehat{V}(2)$. Note that GOE(v^2) means GOE with variance v^2 for the off-diagonal matrix elements and $2v^2$ for the diagonal matrix elements; see Eq. (1.2.4). The mean-field one-body Hamiltonian $\widehat{h}(1)$ in Eq. (1.2.5), in our studies, is a fixed one-body operator defined by the sp energies ϵ_i with average spacing Δ. It is important to note that the operators $h(1)$ and $V(2)$ are independent. Without loss of generality, we put $\Delta = 1$ so that λ, the strength of the interaction, is in the units of Δ. Thus, EGOE(1+2) is defined by the three parameters (N, m, λ). It is possible to draw the ϵ_i's from the eigenvalues of a random ensemble and then the corresponding EGOE(1+2) is called two-body random interaction model (TBRIM) [Fl-97] or from the center of a GOE and then the corresponding EGOE(1+2) is called random interaction matrix model (RIMM) [Al-00,Al-01]. Construction of the EGOE(1+2) ensemble in m-fermion spaces follows easily from the results in Sec. 1.2.1. The notation used in Eq. (1.2.5) implies that the action of the operator $\{\widehat{H}\}$ on the m-particle basis space generates EGOE(1+2) Hamiltonian matrix ensemble in m-particle spaces. It should be noted that the embedding for EGOE(1+2) is also generated by the $SU(N)$ group and the propagation formulas for the energy centroids and variances of the eigenvalue densities follow from the unitary decomposition of H with respect to the $U(N)$ algebra; see Appendix A.

The most significant aspect of EGOE(1+2) is that the ensemble admits three chaos markers as λ is increased form zero. Firstly, eigenvalue (state) densities approach Gaussian form for large m for all values of λ. As the value of λ increases from zero, level fluctuations exhibit transition from Poisson to GOE at $\lambda = \lambda_c$ [Ja-97]. With further increase in the λ value, strength functions (also called local density of states) make a transition from Breit-Wigner (BW) to Gaussian form at $\lambda = \lambda_F >> \lambda_c$ [Ge-97, Ko-01a, Ja-02]. Beyond this point, there is a region around $\lambda = \lambda_d$ where entropies and other statistics become same in the eigenbasis of the mean-field Hamiltonian and the pure two-body Hamiltonian [Ho-95, Ko-02] (though not yet proved, this result perhaps extends to any basis [Ko-03]). Equivalently, all different definitions for thermodynamic properties like entropy, temperature etc. coincide at $\lambda = \lambda_d$. It should be stressed that the chaos markers form the basis [Ko-03] for statistical spectroscopy [Ko-01, Fr-82, Ka-94, Fl-99, Fr-06, Ko-10]. The parametric dependence of λ_c, λ_F and

λ_d is also known and this will be discussed in detail in Chapter 2. Besides these, generic properties of EGOE(2) are valid for EGOE(1+2) in the strong coupling limit; i.e., for $\lambda >> \lambda_F$. Detailed discussion of the three chaos markers $(\lambda_c, \lambda_F, \lambda_d)$ generated by EGOE(1+2) and also applications of the ensemble are given in [Ko-01, Ko-03, An-04, Br-08, Go-11] and references therein. Now, we will turn to embedded Gaussian unitary ensembles for spinless fermions.

1.2.3 EGUE(2) and EGUE(k) ensembles

For a system of m fermions occupying N sp states, all the $N_m = d_f(N,m) = \binom{N}{m}$ antisymmetric states transform as the irreducible representation (irrep) $f_m = \{1^m\}$, in Young tableaux notation, with respect to the $U(N)$ algebra. With only two-body interactions among the fermions, the Hamiltonian operator is

$$\widehat{H} = \sum_{\nu_a,\nu_b} V_{\nu_a\,\nu_b}(2) A^\dagger(f_2\nu_b) A(f_2\nu_a)\,. \tag{1.2.6}$$

Here, $f_2 = \{1^2\}$ and ν_r's denote irreps of the groups in the subgroup chain of $U(N)$ that supply the labels for a complete specification of any two-particle state; similarly, for any m, the states are $|f_m\nu_m\rangle$. Note that $A^\dagger$ and A in Eq. (1.2.6) are normalized two-particle creation and destruction operators, respectively and $V_{\nu_a\,\nu_b}(2)$ are two-particle matrix elements. The EGUE(2) ensemble in m-particle spaces, with matrix dimension $N_m = d_f(N,m)$, is generated by the $\widehat{H}$ operator in Eq. (1.2.6) with GUE representation in two-particle spaces and then propagating it to the m-particle spaces using the direct product structure of the m-particle states [Ko-05]. With the two-particle matrix elements $V_{\nu_a\,\nu_b}(2)$ [the $V(2)$ matrix being complex hermitian] drawn from a GUE, $V_{\nu_a\,\nu_b}(2)$ are independent Gaussian variables with zero center and variance given by,

$$\overline{V_{\nu_a\,\nu_b}(2)\,V_{\nu_c\,\nu_d}(2)} = \lambda^2 \delta_{\nu_a\nu_d}\delta_{\nu_b\nu_c}\,. \tag{1.2.7}$$

Here, $\lambda^2 N_2$ is the ensemble averaged two-particle variance. As in [Ko-05], the $U(\Omega) \leftrightarrow SU(\Omega)$ correspondence is used and therefore we use $U(\Omega)$ and $SU(\Omega)$ interchangeably when there is no confusion. Important step in the analytical study of EGUE(2) is the unitary decomposition of $\widehat{H}$ in terms of the $SU(N)$ tensors $B(g_\nu\omega_\nu)$

with $g_\nu = \{0\}, \{21^{N-2}\}$ and $\{2^2 1^{N-4}\}$,

$$B(g_\nu \omega_\nu) = \sum_{v_a, v_b} \left\langle f_2 v_a \overline{f_2} \overline{v_b} \mid g_\nu \omega_\nu \right\rangle A^\dagger(f_2 v_a) A(f_2 v_b) \,, \tag{1.2.8}$$

where $\overline{f}$ is the irrep conjugate to f and $\left\langle f_2 v_a \overline{f_2} \overline{v_b} \mid g_\nu \omega_\nu \right\rangle$ is a $SU(N)$ Wigner coefficient. For $f_2 = \{1^2\}$ we have, $\overline{f_2} = \{1^{N-2}\}$ and it also contains a phase factor as discussed in [Ko-05]. Then we have $\widehat{H} = \sum_{g_\nu, \omega_\nu} W(g_\nu \omega_\nu) B(g_\nu \omega_\nu)$. A significant property of the expansion coefficients W's is that they are also independent Gaussian random variables, just as V's, with zero center and variance given by $\overline{W(g_\nu \omega_\nu) W(g_\mu \omega_\mu)} = \lambda^2 \delta_{g_\nu g_\mu} \delta_{\omega_\nu \omega_\mu}$. Using Wigner-Eckart theorem, the matrix elements of B's in f_m space can be decomposed into a reduced matrix element and a $SU(\Omega)$ Wigner coefficient. Using this and the expansion of $\widehat{H}$ in terms of B's, exact analytical formulas are derived for the ensemble averaged spectral variances, cross-correlations in energy centroids and also for the cross-correlations in spectral variances for EGUE(2) [Ko-05]. In addition, exact result for the ensemble averaged excess parameter (this involves fourth moment) for the density of eigenvalues is also derived [Ko-05]. An alternative derivation was given by Benet et al [Be-01a, Be-01b]. More significantly, all these results extend to EGUE(k); i.e., EGUE generated by k-body interactions. Two significant results for EGUE(k) are: (i) for $k \le m < 2k$, the density of eigenvalues is semi-circular whereas the density is Gaussian for $m >> 2k$ with $m = 2k$ being the transition point; (ii) EGUE(k) generates non-zero cross-correlations between states with different particle numbers while they will be zero for GUE representation for the m-particle H matrices. See [Ko-05, Be-01a, Be-01b, Pl-02, Ko-06a] for further details; cross-correlations are defined and further explored in Chapters 4 and 6 ahead.

1.3 Embedded Ensembles for Spinless Boson Systems

The BEGOE(2)/BEGUE(2) ensemble for spinless boson systems is generated by defining the two-body Hamiltonian H to be GOE/GUE in two-particle spaces and then propagating it to many-particle spaces by using the geometry of the many-particle spaces [this is in general valid for k-body Hamiltonians, with $k < m$, generating BEGOE(k)/BEGUE(k)]. Consider a system of m spinless bosons occupying N sp states $|\nu_i\rangle$, $i = 1, 2, \ldots, N$; see Fig. 1.5. Then, BEGOE(2) is defined by the Hamilto-

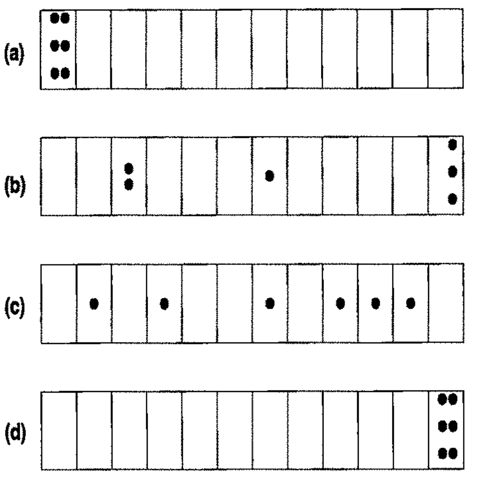

Figure 1.5: Figure showing some configurations for the distribution of $m=6$ spinless bosons in $N=12$ single particle states. Distributing the m bosons in all possible ways in N single particle states generates the m-particle configurations or basis states. This is similar to distributing m particles in N boxes with the conditions that occupancy of each box lies between zero and m and the maximum number of occupied boxes equals m. In the figure, (a) corresponds to the basis state $\left|(\nu_1)^6\right\rangle$, (b) corresponds to the basis state $\left|(\nu_3)^2\nu_7(\nu_{12})^3\right\rangle$, (c) corresponds to the basis state $|\nu_2\nu_4\nu_7\nu_9\nu_{10}\nu_{11}\rangle$ and (d) corresponds to the basis state $\left|(\nu_{12})^6\right\rangle$.

nian operator,

$$\widehat{H}=\sum_{\nu_i\leq\nu_j,\,\nu_k\leq\nu_l}\frac{\langle\nu_k\,\nu_l\mid\widehat{H}\mid\nu_i\,\nu_j\rangle}{\sqrt{(1+\delta_{ij})(1+\delta_{kl})}}\,b^\dagger_{\nu_k}\,b^\dagger_{\nu_l}\,b_{\nu_i}\,b_{\nu_j}\,, \tag{1.3.1}$$

with the symmetries for the symmetrized two-body matrix elements $\langle\nu_k\,\nu_l\mid\widehat{H}\mid\nu_i\,\nu_j\rangle$ being,

$$\begin{aligned}\langle\nu_k\,\nu_l\mid\widehat{H}\mid\nu_j\,\nu_i\rangle&=\langle\nu_k\,\nu_l\mid\widehat{H}\mid\nu_i\,\nu_j\rangle\,,\\ \langle\nu_k\,\nu_l\mid\widehat{H}\mid\nu_i\,\nu_j\rangle&=\langle\nu_i\,\nu_j\mid\widehat{H}\mid\nu_k\,\nu_l\rangle\,.\end{aligned} \tag{1.3.2}$$

The action of the Hamiltonian operator defined by Eq. (1.3.1) on an appropriately chosen basis states (see Fig. 1.5 for examples) generates the BEGOE(2) ensemble. Note that b_{ν_i} and $b^\dagger_{\nu_i}$ in Eq. (1.3.1) annihilate and create a boson in the sp state $|\nu_i\rangle$, respectively. The Hamiltonian matrix $H(m)$ in m-particle spaces contains three different types of non-zero matrix elements and explicit formulas for these are [Pa-00],

$$\left\langle\prod_{r=i,j,\ldots}(\nu_r)^{n_r}\mid\widehat{H}\mid\prod_{r=i,j,\ldots}(\nu_r)^{n_r}\right\rangle=\sum_{i\geq j}\frac{n_i\left(n_j-\delta_{ij}\right)}{\left(1+\delta_{ij}\right)}\,\langle\nu_i\nu_j\mid\widehat{H}\mid\nu_i\nu_j\rangle\,,$$

tion of the energy evolution of various observables into a smoothed and a fluctuating part (with fluctuations following GOE/GUE/GSE) provides the basis for statistical spectroscopy [Ko-10, Ko-89, Fr-82]. In statistical spectroscopy, methods are developed to determine various moments defining the distributions (predicted by EGEs) for the smoothed parts (valid in the chaotic region) without recourse to many-body Hamiltonian construction [this part of statistical spectroscopy is also often refered to as spectral distribution theory or spectral distribution methods].

The aim of the present thesis is to identify and systematically analyze many different physically relevant EGEs with symmetries by considering a variety of quantities and measures that are important for finite interacting quantum systems. Numerical as well analytical study of these more general ensembles is challenging due to the complexities of group theory and also due to large matrix dimensions for $m \geq 10$. It is useful to mention that many diversified methods like numerical Monte-Carlo methods, binary correlation approximation, trace propagation, group theory and perturbation theory are used to derive generic properties of EE [Mo-75, Be-01a, Ko-05, Ko-07, Pa-11]. Towards this end, we have obtained large number of new results for embedded ensembles and in particular for EGOE(1+2)-**s**, EGUE(2)-$SU(4)$, EGOE(1+2)-π, BEGOE(1+2)-**s** and EGOE(2)-J ensembles. In addition, derived are formulas for several fourth order traces that are needed in the analysis of EGOE's and also in the applications of spectral distribution theory generated by EGEs. We have also obtained further evidence for EGOE representation of nuclear Hamiltonian matrices. Results of the present thesis together with earlier investigations establish that embedded Gaussian ensembles can be used gainfully to study a variety of problems in many-body quantum physics and this includes some of the new areas of research in physics such as quantum information science (QIS) and the thermodynamics of isolated finite interacting quantum systems. It should be noted that some of the work in the present thesis is also reviewed in [Go-11].

Before going further, for completeness, we mention that, besides the EE(BEE)'s that will be described in detail in Chapters 2-8, there a few other EE(BEE)'s that have received limited attention in the literature. They are: (i) EGOE invariant under particle-hole symmetry, called random quasi-particle ensembles [Jo-98, Ki-07], (ii) a fixed Hamiltonian plus EGOE called K+EGOE [Ko-01] and similarly displaced

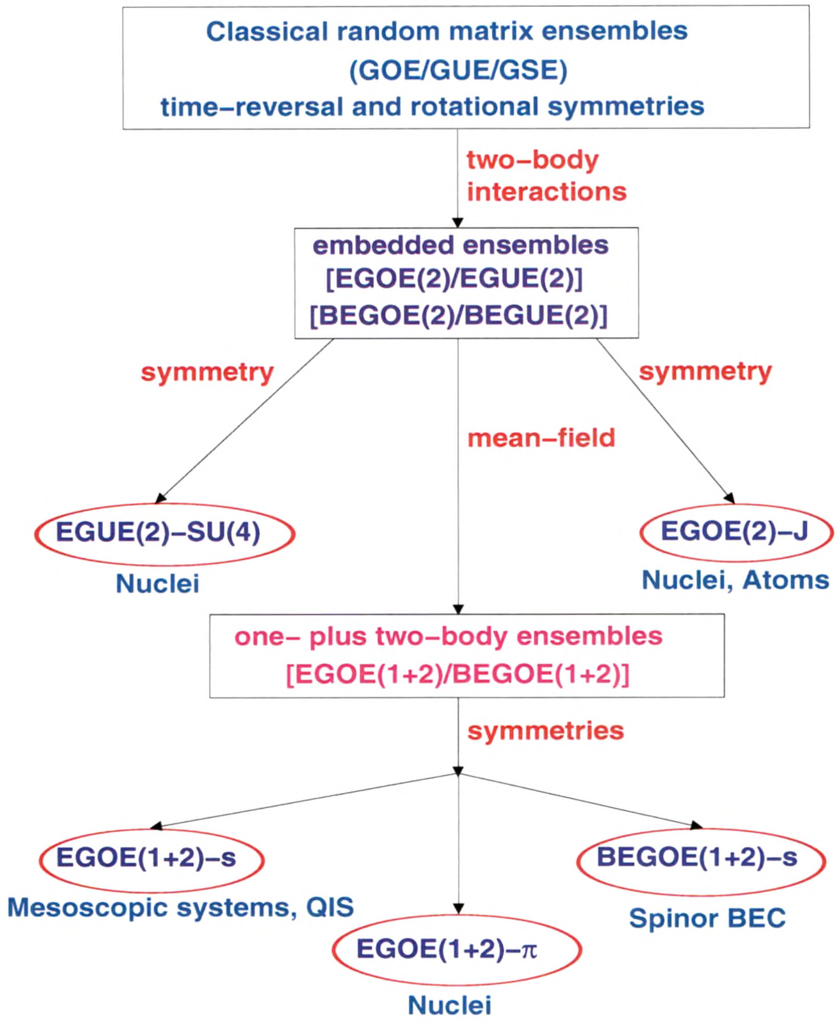

Figure 1.6: Figure showing the information content of various random matrix ensembles. Also shown are the areas in which embedded ensembles with various symmetries are relevant. Here, **s** denotes two-particle spin, $SU(4)$ denotes spin-isospin supermultiplet symmetry, π denotes parity and J denotes total angular-momentum. Note that the symplectic ensembles EGSE/BEGSE and the one plus two-body unitary ensembles EGUE(1+2)/BEGUE(1+2) are not shown as there are no studies of these ensembles till today.

TBRE [Ve-02, Co-82] where a constant is added to all the two-body interaction matrix elements, (iii) EGOE with a partitioned GOE in the two-particle spaces, called p-EGOE [Ko-99, Fr-83], (iv) in mesoscopic systems such as quantum dots, randomness of the sp states induces randomness in the two-body part of the Hamiltonian and these then give rise to induced-TBRE depending on the underlying space-time symmetries as well as on the features of the two-body interaction [Al-05], and (v) BEGOE(1+2) with orbital angular-momentum L, denoted as BEGOE(1+2)-L or BTRBE-L, for bosons in sp orbits [Ku-00] and sd orbits [Bi-01] and also BEGOE(2) with

$SO(N_1) \oplus SO(N_2)$ symmetry in IBM [Ko-04]. Now we will give a preview of the thesis.

Results for transitions in eigenvalue and wavefunction structure in one plus two-body random matrix ensembles with spin [EGOE(1+2)-**s**] are given in Chapter 2. Chapter 3 gives the results for pairing correlations generated by EGOE(1+2)-**s**. Spectral properties of embedded Gaussian unitary ensemble of random matrices generated by two-body interactions with Wigner's $SU(4)$ symmetry [EGUE(2)-$SU(4)$] are derived and discussed in Chapter 4. It is important to mention that EGUE(2)-$SU(4)$ is introduced for the first time in the present thesis. Chapter 5 gives the results for density of states and parity ratios for one plus two-body random matrix ensembles with parity [EGOE(1+2)-π]. Spectral properties of one plus two-body random matrix ensembles for boson systems with spin [BEGOE(1+2)-**s**] are presented in Chapter 6. Although BEGOE(1+2)-**s** was known in literature before, it is explored in detail for the first time in this thesis. Chapter 7 gives the results for higher order averages, derived by extending Mon and French's binary correlation method to two-orbits, required in many applications. In addition, given also are some results for the traces needed for the embedded Gaussian orthogonal ensemble of two-body interactions with angular-momentum J symmetry [EGOE(2)-J] for fermions in a single j-shell. Chapter 8 gives a comprehensive analysis of the structure of H matrices to establish EGOE structure of nuclear shell model H-matrices. Finally, Chapter 9 gives conclusions and future outlook. Before turning to Chapter 2, we would like to add that there will be some unavoidable repetition in Chapters 2-8 as they deal with different embedded ensembles with applications in different physical systems.