Chapter 5

Trajectory Controllability of Second-order Impulsive Systems

Contents

In this chapter, the authors have examined the trajectory controllability (TC) of second-order evolution systems while taking impulses into account. The cosine family of operators produced by the linear component of the system, the integral version of Gronwall's inequality, and the idea of nonlinear functional analysis were used to describe the TC findings. Applications for both finite-dimensional and infinite-dimensional systems of the TC-controlled systems are given.

5.1 Introduction

The ability to be controlled is a fundamental qualitative quality of systems. In order to move a system from a given starting state to a desired end state or one that is near it, one must identify the controllers that will do it. Kalmann was the first to introduce the notion of controllability using the concept of functional analysis. The monographs [125, 140, 22], and articles [70, 82, 80, 90, 51, 130, 72, 45] and reference their work discuss the study of various types of controllabilities for the linear, nonlinear systems using functional analytic approach.

One should determine the controller that moves the system from the supplied beginning state to the desired final state in order to examine different types of controllability of the system. This style of controller, meanwhile, could not be cost-effective. Hence, George [122] introduced the Trajectory Controllability (TC) problems. Instead of leading the system from a specific starting condition to the intended end state, the challenge has been to build a control that directs it along a preset course. When launching a rocket into space, a specific path and the intended location are necessary for cost-effectiveness. Consequently, TC is explored by numerous researchers [27, 137, 133]. The authors in [133] investigated the TC of a semi-linear parabolic system.

On the other hand, in many systems, the state abruptly changes at a specific moment in time or a small time-period. These systems might be referred to as instantaneous impulsive systems or noninstantaneous impulsive systems. Applications and characteristics of these systems are discussed in [135, 74, 156, 129, 121] and reference therein. Shah *et al.* discussed the TC of a first-order non-instantaneous impulsive system on the Banach space in 2021, [136].

Many evolution systems representing wave phenomena are modeled into second-order systems. Therefore, in this article, the authors have discussed the TC of second order system

$$\begin{cases} x''(t) = \mathcal{A}x + \mathcal{F}(t, x(t), x'(t)) + \varpi(t), \\ x(0) = x_{10}, \quad x'(0) = x_{20}, \end{cases} \tag{5.1.1}$$

by considering non-instantaneous impulses into account over the finite time interval $\Omega = [0, T_0]$. Here, at each time t, the state lies in $\mathbb{X}$, $\mathcal{A}$ is the linear on $\mathbb{X}$, $\mathcal{F} : \Omega \times \mathbb{X}^3 \to \mathbb{X}$ is a non-linear function, and $\varpi(t)$ is the trajectory controller of the system.

5.2 Preliminaries

Definition 5.2.1 (Complete Controllability). *[136] "The evolution system completely controllable on the interval $\Omega = [0, T_0]$ if for any $x_0, x_1 \in \mathbb{X}$ there exists a controller $\varpi(t)$ in the control space $\mathbb{U}$ such that the state of system steers initial state x_0 at $t = 0$ to desire final state x_1 at $t = T$."*

Definition 5.2.2 (Total Controllability). *[136] "The evolution system is totally controllable on the interval $\Omega = [0, T_0]$ if it is completely controllable over all its subintervals $[t_k, t_{k+1}]$."*

Let $\mathcal{C}_\Omega$ be the set of all functions $\acute{x}(\cdot)$ defined over Ω satisfying the initial state and final state $x(0) = x_0$ and $\acute{x}(T) = x_1$, respectively. This set $\mathcal{C}_\Omega$ is called the set of all feasible trajectories. The controller obtained from the concept of complete and total controllability for the linear system will be optimal but for the semi-linear or nonlinear system may not be optimal. To overcome this situation one has to design a trajectory having optimum energy or cost and define a controller in such a way that the state of the system steers along this trajectory. Finding the controller that steers the system on the prescribed optimal trajectory from an initial state to the desired final state is called TC.

Definition 5.2.3 (TC). *[136] "The evolution system is trajectory controllable (T-Controllable) if for any trajectory $\acute{x} \in \mathcal{C}_\mathcal{T}$, there exists L^2 control function $w \in \mathbb{U}$ such that the state of the system $x(t)$ satisfy $x(t) = \acute{x}(t)$ almost everywhere over Ω."*

In TC, one must identify the controller that will cause the system to steer along a predetermined course or trajectory from an arbitrary beginning state to the desired final state. Consequently, TC is the most powerful kind of controllability.

5.3 TC without Impulses

In this section, the authors have discussed the TC of the second-order system

$$\begin{cases} x''(t) = \mathcal{A}x + \mathcal{F}(t, x(t), x'(t)) + \varpi(t), \\ x(0) = x_{10}, \quad x'(0) = x_{20}, \end{cases} \tag{5.3.1}$$

without considering the impulses over Ω. Assuming $\mathcal{F}$ good enough to have a unique mild solution

$$x(t) = \mathcal{C}(t)x_{10} + \mathcal{S}(t)x_{20} + \int_0^t \mathcal{S}(t-\tau)[\mathcal{F}(\tau, x(\tau), x'(\tau)) + \varpi(\tau)]\,\mathrm{d}\tau, \tag{5.3.2}$$

for all $t \in \Omega$ and any measurable function $\varpi(t)$. Where $\mathcal{C}(\cdot)$ is a strongly continuous cosine family of operators generated by the linear part $\mathcal{A}$ and $\mathcal{S}(\cdot)$ is associated sine family of operators.

Theorem 5.3.1. *The system* (5.3.1) *is Trajectory controllable over Ω if $\mathcal{F}$ is measurable with t, continuous with respect to other arguments, and there exist positive constants L_{F1} and L_{F2} such that*

$$\|\mathcal{F}(t, x_1, \acute{x}_1) - \mathcal{F}(t, x_2, \acute{x}_2)\| \leq L_{F1}\|x_1 - x_2\| + L_{F2}\|\acute{x}_1 - \acute{x}_2\|.$$

Proof. Let $u(t)$ be any trajectory from $\mathcal{C}_\Omega$ which steers the evolution equation (5.3.1) from the initial state x_{10} to desired final state x_{11}. Define trajectory controller $\varpi(t)$ as

$$\varpi(t) = u''(t) - \mathcal{A}u(t) + \mathcal{F}(t, u(t), u'(t)), \tag{5.3.3}$$

and plugging it in the system (5.3.1), the system becomes:

$$x''(t) = \mathcal{A}x + \mathcal{F}(t, x(t), x'(t)) + u''(t) - \mathcal{A}u(t) + \mathcal{F}(t, u(t), u'(t)). \tag{5.3.4}$$

Considering $\mathfrak{z}(t) = x(t) - u(t)$, the system (5.3.4) becomes

$$\mathfrak{z}''(t) = \mathcal{A}\mathfrak{z}(t) + \mathcal{F}(t, x(t), x'(t)) - \mathcal{F}(t, u(t), u'(t)), \tag{5.3.5}$$

with conditions $\mathfrak{z}(0) = 0$, $\mathfrak{z}'(0) = 0$, and the mild solution of the system (5.3.5) satisfies

$$\|\mathfrak{z}(t)\| \leq \int_0^t \|\mathcal{S}(t-\tau)\| \; \|\mathcal{F}(\tau, x(\tau), x'(\tau)) - \mathcal{F}(t, u(t), u'(t))\| \; \mathrm{d}\tau.$$

Assuming properties of strongly continuous cosine family of the operators generated by linear part $\mathcal{A}$ and the hypotheses of the theorem,

$$\begin{aligned} \|\mathfrak{z}(t)\| &\leq \int_0^t \|\mathcal{S}(t-\tau)\| \, L_{F1} \; \|x(\tau) - u(\tau)\| + L_{F2} \; \|\mathfrak{x}'(\tau) - u'(\tau)\| \; \mathrm{d}\tau \\ &\leq K \int_0^t \left(L_{F1} \, \|\mathfrak{z}(\tau)\| + L_{F2} \, \|\mathfrak{z}'(\tau)\|\right) \mathrm{d}\tau, \qquad K = \|S(\cdot)\|. \end{aligned}$$

Differentiating ($\|\mathfrak{z}(t)\|$ is differentiable a.e) the above inequality

$$\|\mathfrak{z}'(t)\| \leq K \left(L_{F1} \|\mathfrak{z}(t)\| + L_{F2} \|\mathfrak{z}'(t)\|\right),$$

simplifying

$$\|\mathfrak{z}'(t)\| \leq \frac{K L_{F1}}{1 - K L_{F2}} \|\mathfrak{z}(t)\|.$$

Applying a differential form of Grönwall's inequality $\|\mathfrak{z}(t)\| = 0$ a.e., and thus $x(t) = u(t)$ a.e. Hence the system (5.3.1) is trajectory controllable over Ω. □

Example 5.3.1. *The equations of motion for the artificial satellite due to the oblateness of the earth are modeled into second-order equations*

$$\begin{aligned} \mathfrak{x}''(t) &= -\frac{\mu}{r^3}\mathfrak{x} - \frac{3\mu R^2 J_2 \mathfrak{x}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2r^7}, \\ \mathfrak{y}''(t) &= -\frac{\mu}{r^3}\mathfrak{y} - \frac{3\mu R^2 J_2 \mathfrak{y}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2r^7}, \\ \mathfrak{z}''(t) &= -\frac{\mu}{r^3}\mathfrak{z} - \frac{3\mu R^2 J_2 \mathfrak{y}(3\mathfrak{x}^2 + 3\mathfrak{y}^2 - 2\mathfrak{z}^2)}{2r^7}, \end{aligned} \tag{5.3.6}$$

where G is the universal gravitational constant, R, M are radius, mass of earth, $\mu = GM$, J_2 is zonal coefficient, and $r = \sqrt{\mathfrak{x}^2 + \mathfrak{y}^2 + \mathfrak{z}^2}$. From the various studies, it was found that the motion of the artificial satellite is unstable under the oblate earth if the initial velocity is low and sometimes it can hit the surface of the earth [138]. Therefore to make the motion in the prescribed orbit one has to plug the controller into the satellite so that they follow a specific path. Let $[u_1(t), u_2(t), u_3(t)]$ and $[w_1(t), w_2(t), w_3(t)]$ be the prescribed trajectory and the trajectory controller for *the satellite, respectively. Plugging it in* (5.3.6)*, the equations of motion becomes:*

$$\begin{aligned}
\mathfrak{x}''(t) &= -\frac{\mu}{r^3}\mathfrak{x} - \frac{3\mu R^2 J_2 \mathfrak{x}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2r^7} + w_1(t),\\
\mathfrak{y}''(t) &= -\frac{\mu}{r^3}\mathfrak{y} - \frac{3\mu R^2 J_2 \mathfrak{y}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2r^7} + w_2(t),\\
\mathfrak{z}''(t) &= -\frac{\mu}{r^3}\mathfrak{z} - \frac{3\mu R^2 J_2 \mathfrak{y}(3\mathfrak{x}^2 + 3\mathfrak{y}^2 - 2\mathfrak{z}^2)}{2r^7} + w_3(t).
\end{aligned} \tag{5.3.7}$$

Since the motion of many low earth satellites has a circular orbit having fixed radius $r = a$ from the center of the earth. Therefore, the equation of motion for the circular orbit $r = a$ becomes:

$$\begin{aligned}
\mathfrak{x}''(t) &= -\frac{\mu}{a^3}\mathfrak{x} - \frac{3\mu R^2 J_2 \mathfrak{x}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2a^7} + w_1(t),\\
\mathfrak{y}''(t) &= -\frac{\mu}{a^3}\mathfrak{y} - \frac{3\mu R^2 J_2 \mathfrak{y}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2a^7} + w_2(t)\\
\mathfrak{z}''(t) &= -\frac{\mu}{a^3}\mathfrak{z} - \frac{3\mu R^2 J_2 \mathfrak{y}(3\mathfrak{x}^2 + 3\mathfrak{y}^2 - 2\mathfrak{z}^2)}{2a^7} + w_3(t).
\end{aligned} \tag{5.3.8}$$

These motion equations have the following form

$$\bar{r}''(t) = \mathcal{A}\bar{r}(t) + \mathcal{F}(t, \bar{r}(t)) + \varpi(t), \tag{5.3.9}$$

where, $\bar{r} = [\mathfrak{x}(t), \mathfrak{y}(t), \mathfrak{z}(t)]$ the position vector of the satellite,

$$\mathcal{A} = \begin{bmatrix} -\frac{\mu}{a^3} & 0 & 0 \\ 0 & -\frac{\mu}{a^3} & 0 \\ 0 & 0 & -\frac{\mu}{a^3} \end{bmatrix}, \qquad \mathcal{F}(\bar{r}(t)) = \begin{bmatrix} -\frac{3\mu R^2 J_2 \mathfrak{x}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2a^7} \\ -\frac{3\mu R^2 J_2 \mathfrak{y}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2a^7} \\ -\frac{3\mu R^2 J_2 \mathfrak{y}(3\mathfrak{x}^2 + 3\mathfrak{y}^2 - 2\mathfrak{z}^2)}{2a^7} \end{bmatrix}.$$

The function $\mathcal{F}(\bar{r}(t))$ is differentiable with respect to $\bar{r}$ as all of its partial derivatives exist and are continuous over any finite time interval. The linear operator $\mathcal{A}$ generates a strongly continuous cosine family of operators

$$\mathcal{C}(t) = \begin{bmatrix} \cos\sqrt{\frac{\mu}{a^3}}t & 0 & 0 \\ 0 & \cos\sqrt{\frac{\mu}{a^3}}t & 0 \\ 0 & 0 & \cos\sqrt{\frac{\mu}{a^3}}t \end{bmatrix},$$

and associated sine family

$$\mathcal{S}(t) = \sqrt{\frac{a^3}{\mu}} \begin{bmatrix} \sin\sqrt{\frac{\mu}{a^3}}t & 0 & 0 \\ 0 & \sin\sqrt{\frac{\mu}{a^3}}t & 0 \\ 0 & 0 & \sin\sqrt{\frac{\mu}{a^3}}t \end{bmatrix}.$$

Thus the motion of the satellite (5.3.9) *is trajectory controllable for the finite time intervals. Let the initial position of the satellite be*

$$\bar{r}_0 = \big[0,\ -5888.9727,\ -3400\big],$$

having initial velocity $\bar{v}_0 = [7, 0, 0]$*. Figure 5.1 shows that the motion of the satellite is not stable without a controller. Data:*

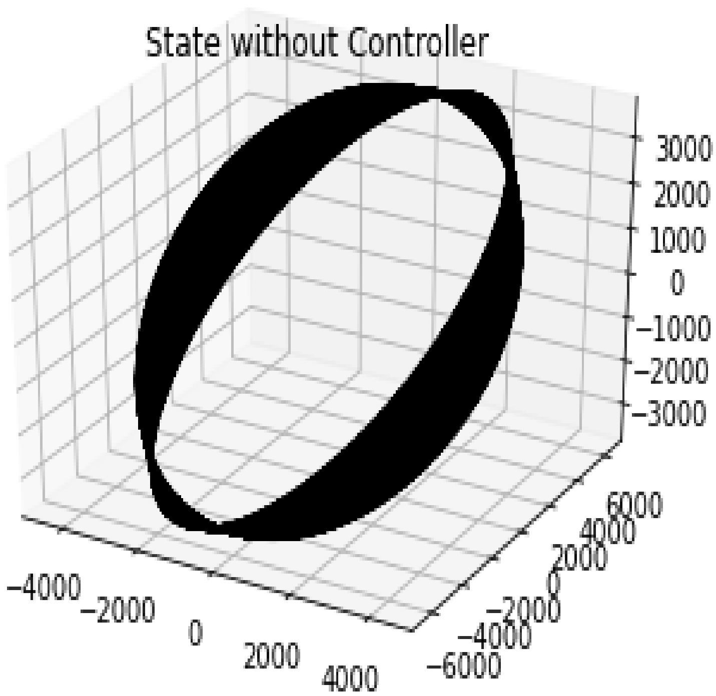

Figure 5.1: The motion of the satellite is not stable without a controller.

$$\bar{r_0} = (0, -5888.9727, -3400), \quad \bar{v_0} = (7, 0, 0), \quad R = 6378.1363, \quad a = |\bar{r_0}|,$$

$\mu = G * M$, $J_2 = 1082.63 \times 10^{-6}$, *Time Span: 540000 sec. Now considering the trajectory for the motion of the satellite*

$$u(t) = \left[7\sqrt{\frac{a^3}{\mu}} \sin\sqrt{\frac{\mu}{a^3}}t \quad -5888.9727\cos\sqrt{\frac{\mu}{a^3}}t \quad -3400\cos\sqrt{\frac{\mu}{a^3}}t \right],$$

and define the trajectory controller $\varpi(t) = \bar{u}'' - \mathcal{A}\bar{u} - \mathcal{F}(\bar{u})$ *and plugging into the equation of motion* (3.4.4) *the state of the system follows prescribed path. Figures 5.2 and 5.3 show the trajectory and state after plugging the controller.*

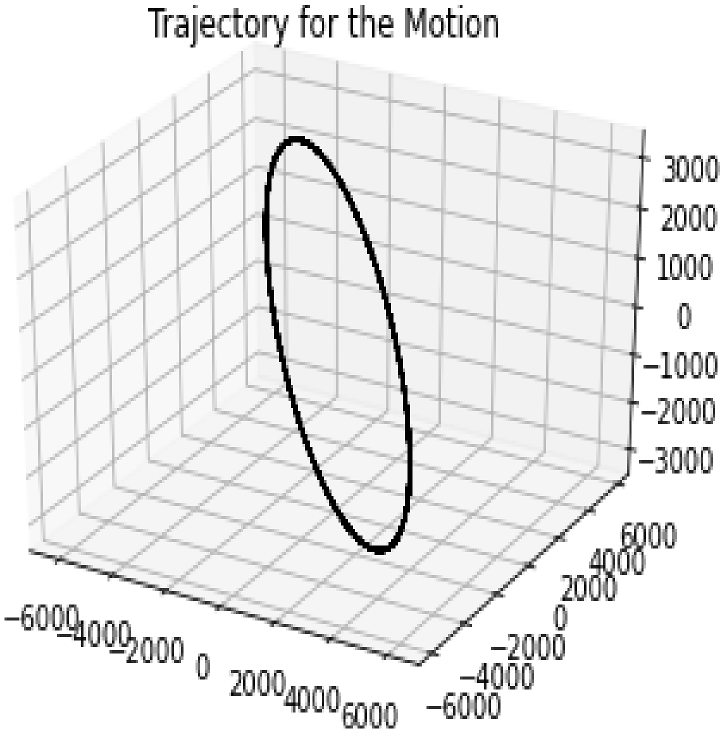

Figure 5.2: Trajectory of System.

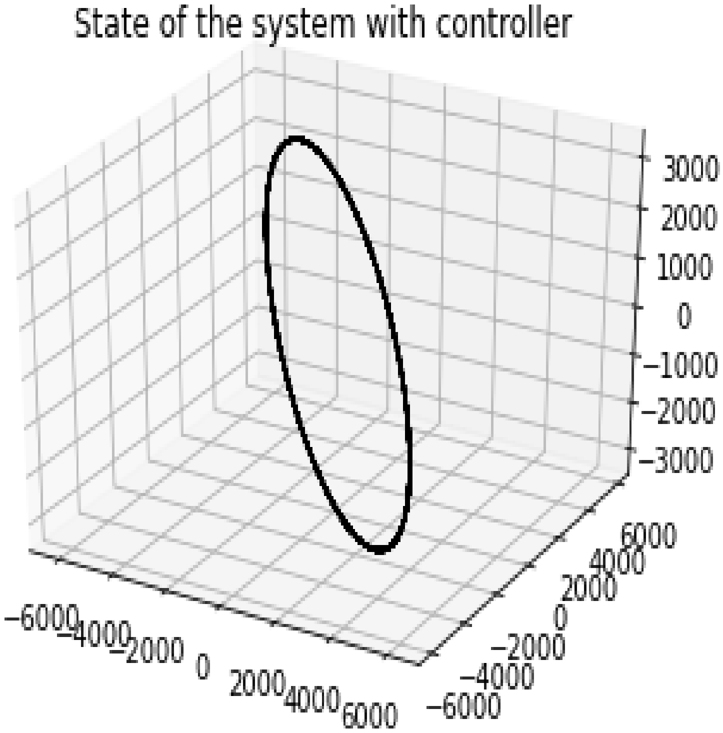

Figure 5.3: state after plugging the controller.

5.4 TC with Impulses

This section discusses the TC of the non-instantaneous impulsive second-order system

$$\begin{cases} x''(t) = \mathcal{A}x + \mathcal{F}(t, x(t), x'(t)) + \varpi(t), & t \in [0, t_1) \cup [s_1, t_2) \cdots \cup [s_\rho, T_0] \\ x(t) = \mathcal{G}_k(t, x(t)) + \varpi_k(t), & t \in [t_1, s_1) \cup [t_2, s_2) \cdots \cup [t_\rho, s_\rho), \\ x(0) = x_{10}, \quad x'(0) = x_{20}, & \end{cases} \tag{5.4.1}$$

over Ω. Attributes of the system (5.4.1) are good enough to have a unique mild solution

$$x(t) = \begin{cases} \mathcal{C}(t)x_{10} + \mathcal{S}(t)x_{20} \\ \qquad + \displaystyle\int_0^t \mathcal{S}(t-\tau)\ [\mathcal{F}(\tau, x(\tau), x'(\tau)) + \varpi(\tau)]\ \mathrm{d}\tau, & t \in [0, t_1), \\ \mathcal{G}_k(t, x(t)) + \varpi_k(t), & t \in [t_k, s_k), \\ \mathcal{C}(t-s_k)\mathcal{G}_k(s_k, x(s_k)) + \mathcal{S}(t-s_k)\mathcal{G}_k'(s_k, x(s_k)) \\ \qquad + \displaystyle\int_{s_k}^t S(t-s)\ [\mathcal{F}(\tau, x(\tau), x'(\tau)) + \varpi(\tau)]\ \mathrm{d}\tau, & t \in [s_k, t_{k+1}), \end{cases} \tag{5.4.2}$$

for all $t \in \Omega$ and any measurable function $\varpi(t)$, where $\mathcal{C}(\cdot)$, $\mathcal{S}(\cdot)$ are a strongly continuous cosine family of operators generated by the linear part $\mathcal{A}$, associated sine family of operators, respectively and $\mathcal{G}_k'$ denote the derivative of $\mathcal{G}_k$ with respect to t.

Assumptions 5.4.1. (A1) *The linear part $\mathcal{A}$ of the equation* (5.4.1)*is an infinitesimal generator of a strongly continuous cosine family of operators;*

(A2) *The nonlinear function $\mathcal{F}$ is measurable with respect to argument t over Ω and there exist constants r_0, L_{F1}, and L_{F2} such that*

$$\|\mathcal{F}(t, x_1, x_2) - \mathcal{F}(t, \acute{x}_1, \acute{x}_2)\| \le L_{F1}\|x_1 - \acute{x}_1\| + L_{F2}\|x_2 - \acute{x}_2\|, \qquad \forall\, x_i, \acute{x}_i \in B_{r_0} \subset \mathbb{X},\ \ i = 1, 2;$$

(A3) *The nonlinear functions $\mathcal{G}_k$ and its time derivative $\mathcal{G}_K'$ for known value of $x(t)$. Moreover there exist $0 < \mathfrak{g}_k < 1$ such that*

$$\|\mathcal{G}_k(t, x) - \mathcal{G}_k(t, \acute{x})\| \le \mathfrak{g}_k\|x - \acute{x}\|, \forall\, x, \acute{x} \in B_{r_0}.$$

Theorem 5.4.1. *The system* (5.4.1) *is trajectory controllable over Ω if hypotheses* (A1)–(A3) *are satisfy.*

Proof. Let $u(t)$ be any trajectory from $\mathcal{C}_\Omega$ which steers the evolution equation (5.4.1) from the initial state x_{10} to desired final state x_{11} satisfying $u(t_k^+) = x(t_k^+)$. Over the interval $[0, t_1)$ the system becomes:

$$x''(t) = \mathcal{A}x + \mathcal{F}(t, x(t), x'(t)) + \varpi(t)x(0) = x_{10}, \qquad x'(0) = x_{20}. \tag{5.4.3}$$

Plugging the controller

$$\varpi(t) = u''(t) - \mathcal{A}u(t) + \mathcal{F}(t, \mathfrak{p}(\bar{\mathfrak{t}}), u'(t)), \tag{5.4.4}$$

in the system (5.4.3), and proceeding in same way as in Theorem 5.3.1, the system is controllable over the interval $[0, t_1)$. Over the interval $[t_k, s_k)$, the system becomes:

$$x(t) = \mathcal{G}_k(t, x(t)) + \varpi_k(t). \tag{5.4.5}$$

Plugging the controller

$$\varpi_k(t) = u(t) - \mathcal{G}_k(t, u(t)), \tag{5.4.6}$$

in the system (5.4.4) the system becomes $x(t) - u(t) = \mathcal{G}_k(t, x(t)) - \mathcal{G}_k(t, u(t))$. Taking $\mathfrak{z}(t) = x(t) - u(t)$ and computing

$$\|\mathfrak{z}(t)\| = \|\mathcal{G}_k(t, x(t)) - \mathcal{G}_k(t, u(t))\| \leq \mathfrak{g}_k \|\mathfrak{z}(t)\|.$$

Thus, $(1 - \mathfrak{g}_k)\|z(t)\| \leq 0$. Since $\mathfrak{g}_k < 1$ therefore $\|z(t)\| = 0$ a.e. Hence, the system is T-controllable over $[t_k, s_k)$, $\forall k = 1, 2, \cdots, \rho$.

Over $[s_k, t_{k+1})$ the system becomes:

$$x''(t) = \mathcal{A}x + \mathcal{F}(t, x(t), x'(t)) + \varpi(t), \tag{5.4.7}$$

with initial conditions $x(s_k) = \mathcal{G}_k(s_k, x(s_k))$ and $x'(s_k) = \mathcal{G}_k'(s_k, x(s_k))$. Since, $\|\mathfrak{z}(t)\| = 0$ for all $t \in [t_k, s_k)$ and continuity of $\mathcal{G}_k$ leads to $\|\mathfrak{z}(s_k)\| = 0$. Thus, $x(s_k) = u(s_k)$. Plugging the controller

$$\varpi(t) = u''(t) - \mathcal{A}u(t) + \mathcal{F}(t, u(t), u'(t)), \tag{5.4.8}$$

in the system (5.4.7) and assuming the hypotheses (A1)-(A3) and using the theorem 5.3.1, the system is trajectory controllable over the interval $[s_k, t_{k+1})$. Hence, the system (5.4.1) is trajectory controllable over Ω. $\square$

Example 5.4.1. *Consider the partial differential equation*

$$\begin{cases} \dfrac{\partial^2 Z(t,x)}{\partial t^2} = Z_{xx}(t,x) + e^{-Z(t,x)} + \varpi(t), & t \in \left[0, \frac{1}{3}\right) \cup \left[\frac{2}{3}, 1\right], \\ Z(t,x) = \dfrac{1}{2}\sin\left(Z(t,x)\right), & t \in \left[\frac{1}{3}, \frac{2}{3}\right), \end{cases} \tag{5.4.9}$$

in the Banach space $\mathbb{X} = L^2(\Omega)$, $\Omega = [0, \pi]$, $T_0 = \pi$, *and with initial condition*

$$Z(0,x) = Z_0(x), \qquad Z_t(0,x) = Z_1(x),$$

and boundary conditions $Z(t,0) = Z(t,\pi) = 0$. *Define an operator* $\mathcal{A}$ *as* $\mathcal{A}Z = Z_{xx}$ *over the domain*

$$\mathrm{Dom}(\mathcal{A}) = \left\{ y \in L^2(\Omega) \ : \ y'' \ \textit{exist and} \ z(0) = z(\pi) = 0 \right\}.$$

The operator $\mathcal{A}$ *represented by*

$$\mathcal{A}z = \sum_{n=1}^{\infty} -n^2 \left\langle z, \sqrt{\frac{2}{\pi}} \sin nx \right\rangle \sqrt{\frac{2}{\pi}} \sin nx, \qquad z \in \mathrm{Dom}(A).$$

The operator $\mathcal{A}$ *is the infinitesimal generator of strongly continuous cosine family* $\mathcal{C}(\cdot)$ *on* $\mathbb{X}$ *defined by*

$$\mathcal{C}(t)z = \sum_{n=1}^{\infty} \cos nt \left\langle z, \sqrt{\frac{2}{\pi}} \sin nx \right\rangle \sqrt{\frac{2}{\pi}} \sin nx,$$

and associated sine family $\mathcal{S}(\cdot)$ *on* $\mathbb{X}$ *defined by*

$$\mathcal{S}(t)z = \sum_{n=1}^{\infty} \frac{1}{n} \sin t \left\langle z, \sqrt{\frac{2}{\pi}} \sin nx \right\rangle \sqrt{\frac{2}{\pi}} \sin nx.$$

The evolution Eq. (5.4.9) *can be formulated as the abstract equation in* $\mathbb{X} = L^2([0,1])$ *as:*

$$\begin{cases} \dfrac{\mathrm{d}^2 \upsilon}{\mathrm{d}t^2} = \mathcal{A}\upsilon(t) + \mathcal{F}(t, \upsilon(t)) + \varpi(t), & t \in \left[0, \frac{1}{3}\right) \cup \left[\frac{2}{3}, 1\right], \\ \upsilon(t) = \mathcal{G}_1(t, \upsilon), & t \in \left[\frac{1}{3}, \frac{2}{3}\right), \\ \upsilon(0) = \upsilon_0, \quad \dfrac{\mathrm{d}\upsilon}{\mathrm{d}t}(0) = 0. \end{cases} \tag{5.4.10}$$

- *The function* $\mathcal{F}(t, \upsilon) = e^{-\upsilon}$ *is continuous function and there exist* $l_{\mathcal{F}}(r) = 1$ *on* B_{r_0} *satisfying*
$$\|\mathcal{F}(t, \upsilon_1) - \mathcal{F}(t, \upsilon_2)\| \leq \|\upsilon_1 - \upsilon_2\|.$$
Thus, by Theorem the system (5.4.10) *is T-controllable over* $[0, 1]$.

- *Assuming that the derivative of* $\frac{1}{2}\sin z$, $z'(t)$ *exist over the interval* $[0, 1]$.

Then, the system (5.4.10) *is T-controllable over* $[0, 1]$.

5.5 Conclusion

In this chapter, the authors have discussed the TC of the second-order systems with and without impulses. The discussion of the TC of the system was obtained using the concept of the cosine family of operators, nonlinear functional analysis, and Grönwall's inequality. Applications to the motion of the artificial satellite and nonlinear one-dimensional wave equations are also added to validate the obtained results.