Chapter 6

Generalized Non-instantaneous Cauchy Problem

Contents

In this chapter, we present the existence of a solution of the fractional-order semi-linear classical and non-local generalized Cauchy problem with non-instantaneous impulses on the Banach space. The existence results of the problem with classical conditions are established using operator semigroup theory and generalized Banach contraction principle. The problem with non-local conditions is established through operator semigroup theory and Krasnoselkii's fixed point theorem. This article also derived unique results for the problem with classical conditions. Finally, illustrations for the Cauchy problem with the classical and nonlocal problems are added to validate the derived results.

6.1 Introduction

In the past few decades, fractional Calculus has become one of the important branches of applied mathematics. This is because fractional-order dynamical models give much better approximations in many physical situations like seepage flow in porous media, anomalous diffusion, the nonlinear oscillations of earthquakes, traffic flow, electromagnetism, and dynamics of many infectious diseases. The details of applications are found in books of [99, 112] and articles of [4, 20, 37, 43, 49, 60, 61, 68, 76, 77, 98, 108, 113, 116, 117, 118, 150]. The existence theory of fractional differential equations and evolution equations of Caputo type with classical conditions using different fixed point theories is found in the articles of [31, 41, 42, 59] and non-local condition is found in the articles of [13, 23, 10, 47, 87, 109, 145, 161].

Changes in state at a fixed moment or for a small interval of time in dynamical systems are modeled into impulsive dynamical systems. These impulses are instantaneous or non-instantaneous depending on the time at which impulses are applied. Existence results and applications of the instantaneous integer order impulsive dynamical or evolution systems are found in [5, 39, 84, 135] while, existence results for the fractional instantaneous impulsive equation are found in [11, 12, 18, 50, 75, 73, 102, 119]. In some dynamic process changes in state are applied for small-time interval time rather than a fixed moment. Existence results of fractional order impulsive dynamical systems and evolution systems with non-instantaneous impulses with local and non-local conditions are studied by [75, 88, 97].

In this chapter, we established sufficient conditions for the existence of the fractional order generalized Cauchy problem:

$$
\begin{aligned}
{}^{c}D^{\lambda}x(t) &= \mathcal{A}s(t) + \mathcal{F}_k k\left(t, x(t), \int_0^t a_k(t,\zeta,x(\zeta))d\zeta\right),\ t \in [s_{k-1}, t_k),\ k = 1, 2, \cdots, p \\
x(t) &= \mathcal{G}_k(k, x(t)),\ t \in [t_{k-1}, t_k)
\end{aligned}
$$

with local condition $x(0) = x_0$ and non-local condition $x(0) = x_0 + h(x)$ over the interval $[0, T_0]$ in a Banach space $\mathbb{X}$. Here $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ is linear operator, $P_k x = \int_0^t a_k(t,\zeta,x(\zeta))d\zeta$ are nonlinear Volterra integral operator on $\mathbb{X}$, $\mathcal{F}_k : [0,T]\times\mathbb{X}\times\mathbb{X} \to \mathbb{X}$ are nonlinear functions applied in the intervals $[\zeta_{k-1}, t_k)$ and $\mathcal{G}_k : [0,T] \times \mathbb{X}$ are set of nonlinear functions applied in the interval $[t_k, \zeta_k)$ for all $k = 1, 2, \cdots, p$.

6.2 Preliminaries

This section discussed preliminaries about fractional differential operators and some definitions and theorems from the functional analysis.

Definition 6.2.1. *[131] "The Liouville-Caputo fractional derivative of order $\beta > 0$, $n-1 < \beta < n$, $n \in \mathbb{N}$, is defined as*

$$^{c}D^{\lambda}_{t_0+}h(t) = \frac{1}{\Gamma(n-\lambda)}\int_{t_0}^{t}(t-q)^{n-\lambda-1}\frac{d^n h(q)}{dq^n}dq$$

where, the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$".

Theorem 6.2.1. *(Banach Fixed Point Theorem)[21] "Let F be closed subset of a Banach Space $(\mathbb{X}, ||\cdot||)$ and let $T : F \to F$ contraction then, T has unique fixed point in F".*

Theorem 6.2.2. *(Krasnoselskii's Fixed Point Theorem)[21] "Let E be closed convex nonempty subset of a Banach Space $(\mathbb{X}, ||\cdot||)$ and P and Q are two operators on E satisfying:*

(1) $Pv + Qs \in E$, whenever $v, s \in E$,

(2) P is contraction,

(3) Q is completely continuous

then, the equation $Pv + Qv = v$ has unique solution".

Definition 6.2.2. *(Completely Continuous Operator)[36] "Let X and Y be Banach spaces. Then the operator $T : D \subset X \to Y$ is called completely continuous if it is continuous and maps any bounded subset of D to a relatively compact subset of Y".*