Chapter 8

Noninstantaneous Impulsive Hilfer Fractional System

Contents

In this chapter, we have considered the non-instantaneous fractional integro-differential evolution system with Hilfer fractional differential operator in the Banach space and discussed its existence results for the mild solution for the equation with local and non-local conditions. These results are obtained by applying the method of a C_0 operator generated by the linear part of the equation combined with the concept of nonlinear functional analysis and the fixed point theorems. We have discussed the examples to highlight the applicability of the results.

8.1 Introduction

Fractional calculus and differential equations became an important branch of applied mathematics. This is because of many problems from the fields of physical sciences, chemical sciences, biological sciences, finance, and image processing which are modeled using fractional differential operators and give better approximations than those modeled using integer order differential operator [2, 99, 112]. Researchers generalized the fractional order differential operator in a way that coincides with the integer order differential operator and this leads to the existence of many differential operators like Riemann-Liouville, Caputo, Grownwell-Letnikov, and Conformable fractional differential operators. The qualitative properties and applications of fractional dynamical systems are found in [41, 42, 155]. Sometimes the system with non-local initial conditions gives better approximations than classical conditions. The qualitative properties and applications of the non-local systems are found in [47]. Hilfer came up with a new fractional differential operator which is a homotopy between Riemann-Liouville and Caputo fractional differential operators. The qualitative properties and applications of dynamical systems with the Hilfer differential operator including classical and non-local conditions are found in [67] and monograph Hilfer (2000)[64].

Systems having an abrupt change in the state at a fixed time moment or in a small time interval are modeled into instantaneous impulsive evolution or not-instantaneous impulsive evolution equation respectively. The qualitative properties and applications of the integer order evolution systems with instantaneous impulses are found in [135] and the same for the fractional systems are found in [75, 73]. In some of the evolutionary processes, not-instantaneous impulses give better approximations instead of instantaneous impulses. The qualitative properties and applications of systems with non-instantaneous impulsive systems are found in [74, 97, 88].

This work considered non-instantaneous impulsive integro-differential fractional order ($0 < \lambda \leq 1$ and $0 \leq \mu \leq 1$) evolution system of Hilfer type

$$\begin{aligned} D_{0+}^{\lambda,\mu} x(t) &= -\mathcal{A}x(t) \\ &\quad + \mathcal{F}\left(t, x(t), \int_0^t a(t,\tau,x(\tau))d\tau\right),\ t \in \left[\cup [s_i, t_{i+1})\right] \cup [s_p, T_0] \\ x(t) &= \mathcal{G}_k(t, x(t)),\ t \in [t_1, s_1) \cup [t_2, s_2) \cup \cdots \cup [t_p, s_p), \end{aligned}$$

and discussed the existence of solutions with local condition $\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}x(0) = x_0$ and non-local $\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}[x(0) - h(x)] = u_0$ initial conditions over the finite interval $[0, T_0]$ in a Banach space $\mathbb{X}$. $D^{\lambda,\mu}$ differential operators of Hilfer type, $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ is a linear part of the integrodifferential evolution equation, $Kx = \int_0^t a(t, \tau, x(\tau))d\tau$ is nonlinear Volterra integral operator on $\mathbb{X}$, $\mathcal{F} : [0, T_0] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is nonlinear function and $\mathcal{G}_k : [0, T_0] \times \mathbb{X}$ are set of non-linear functions applied in the interval $[t_k, s_k)$ for all $i = 1, 2, \cdots, p$.

The outline of this chapter is as follows: Section-8.2 discusses the preliminaries to establish the results, section-8.3 established existence result for the non-instantaneous fractional order Hilfer integro-differential evolution system with classical conditions followed by a nonlocal condition in section-8.4. Finally, the conclusion is discussed in section-8.5.

8.2 Preliminaries

This section is devoted to the definitions of integral $\mathcal{I}_{t_0}^{\lambda}$ and Hilfer fractional differential operator $\mathcal{D}_{t_0+}^{\lambda,\mu}$, Wright-type function M_λ and properties, concept operator semi-group $\mathcal{T}(t)$, the operators like $\mathcal{K}_\lambda(t)$, $\mathcal{S}_{\lambda,\mu}(t)$.

Definition 8.2.1. *([67]) For $\lambda > 0$, the fractional integral of order λ of a function $h(t)$ is defined by*

$$\mathcal{I}_{t_0}^{\lambda}h(t) = \frac{1}{\Gamma(\lambda)} \int_{t_0}^{t} (t - \tau)^{\lambda-1}h(\tau)d\tau,$$

provided the integral on the right exists.

Definition 8.2.2. *[67] The Hilfer fractional derivative of order λ, $0 < \lambda < 1$ and type μ, $0 \leq \mu \leq 1$ is defined by*

$$\mathcal{D}_{t_0+}^{\lambda,\mu}h(t) = \mathcal{I}_{t_0+}^{\mu(1-\lambda)} \frac{d}{dt} \mathcal{I}_{t_0+}^{(1-\lambda)(1-\mu)}h(t),$$

provided the right value exists.

Definition 8.2.3. *[67] For all $\theta \in \mathbb{C}$ and $\lambda > 0$, the Wright-type function M_λ is defined as:*

$$M_\lambda(\theta) = \sum_{n\in\mathbb{N}} \frac{(-\theta)^{n-1}}{\Gamma(1 - \lambda n)(n - 1)!} \tag{8.2.1}$$

provided the sum on the right exists.

Wright-type function satisfies the following properties:

(1) $M_\lambda(\theta) > 0$ for all $\lambda > 0$.
(2) For $-1 < \eta < \infty$ the integral, $\int_0^\infty \theta^\eta M_\lambda(\theta)d\theta = \frac{\Gamma(1+\eta)}{\Gamma(1+\lambda\eta)}$.
(3) For $r > 0$ the integral, $\int_0^\infty \frac{\lambda}{\theta^{\lambda+1}} e^{-r\theta} M_\lambda(\theta^{-\lambda})d\theta = e^{-r^\lambda}$ for all $\lambda > 0$.

Let, $\mathcal{T}(t)$ be the family of semi-group generated by the linear operator $-\mathcal{A}$ and define two linear operators $\mathcal{S}_\lambda(t)$ and $\mathcal{Q}_\lambda(t)$ as:

$$\mathcal{S}_\lambda(t) = \int_0^\infty M_\lambda(\theta)\mathcal{T}(t^\lambda\theta)d\theta \tag{8.2.2}$$

$$\mathcal{Q}_\lambda(t) = \int_0^\infty \lambda\theta M_\lambda(\theta)\mathcal{T}(t^\lambda\theta)d\theta \tag{8.2.3}$$

Following properties are satisfied by $\mathcal{S}_\lambda(t)$ and $\mathcal{Q}_\lambda(t)$.

Lemma 8.2.1. *[67] If $\mathcal{T}(t)$ be the family of C_0-semigroup generated by the linear operator $-A$ for all $t \in [0, T_0]$ then the families of operators $\mathcal{S}_\lambda(t)$ and $\mathcal{Q}_\lambda(t)$ defined by* (9.2.2) *and* (9.2.3) *are:*
(1) continuous and bounded for all $t \in [0, T_0]$.
(2) strongly continuous over the interval $t \in (0, T_0]$.

The operators $\mathcal{S}_\lambda(t)$ and $\mathcal{Q}_\lambda(t)$ generate new linear operators $\mathcal{S}_{\lambda,\mu}(t)$ and $\mathcal{K}_\lambda(t)$.

$$\mathcal{S}_{\lambda,\mu}(t) = \mathcal{I}_0^{\mu(1-\lambda)}\mathcal{K}_\lambda(t) \tag{8.2.4}$$

$$\mathcal{K}_\lambda(t) = t^{\lambda-1}\mathcal{Q}_\lambda t \tag{8.2.5}$$

These operators $\mathcal{S}_{\lambda,\mu}(t)$ and $\mathcal{K}_\lambda(t)$ satisfies following properties:

Lemma 8.2.2. *[67] If $\mathcal{T}(t)$ be the family of C_0-semigroup generated by the linear operator $-A$ for all $t \in [0, T_0]$ then the families of operators $\mathcal{S}_{\lambda,\mu}(t)$ and $\mathcal{K}_\lambda(t)$ defined by* (8.2.4) *and* (8.2.5) *are:*
(1) continuous and bounded for all $t \in [0, T_0]$.
(2) strongly continuous over the interval $t \in (0, T_0]$.
(3) $||\mathcal{K}_\lambda(t)x|| \leq \frac{t^{\lambda-1}M}{\Gamma(\lambda)}||x||$

(4) $||\mathcal{S}_{\lambda,\mu}(t)x|| \leq \frac{M(\mu(1-\lambda))t^{\lambda+\mu-\lambda\mu-1}}{\Gamma(\lambda+\mu-\lambda\mu-1)}||x||$.
Where, M is the bound of $\mathcal{T}t$ over the interval $[0, T_0]$.

Theorem 8.2.1. *(Banach Fixed Point Theorem [21]) Let E be closed subset of a Banach Space* $(\mathbb{X}, ||\cdot||)$ *and let $T : E \to E$ contraction then, T has unique fixed point in E.*

Theorem 8.2.2. *(Krasnoselskii's Fixed Point Theorem [21]) Let E be closed convex nonempty subset of a Banach Space* $(\mathbb{X}, ||\cdot||)$ *and P and Q are two operators on E satisfying:*

(1) $Pu + Qv \in E$, whenever $u, v \in E$,

(2) P is contraction,

(3) Q is completely continuous

then, the equation $Pu + Qu = u$ has a solution.

Operator Semigroup

8.3 Equation with Classical Conditions

This section establishes the existence results for fractional order ($0 < \lambda \leq 1$ and $0 \leq \mu \leq 1$) non-instantaneous impulsive Hilfer integro-differential evolution systems with classical conditions.

$$\begin{aligned} D_{0+}^{\lambda,\mu}x(t) &= -Ax(t) + \mathcal{F}\left(t, x(t), \int_0^t a(t,\tau,x(\tau))d\tau\right), \\ &\qquad t \in \left[\cup [s_i, t_{i+1})\right] \cup [s_p, T_0] \\ x(t) &= \mathcal{G}_k(t, x(t)),\ t \in [t_1, s_1) \cup [t_2, s_2) \cup \cdots [t_p, s_p) \\ \mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}x(0) &= x_0 \end{aligned} \tag{8.3.1}$$

over the interval $[0, T_0]$ in the Banach space $\mathbb{X}$.

Definition 8.3.1. *The function $x(t)$ is called mild solution of the impulsive fractional equation* (8.3.1) *over the interval* $[0, T_0]$ *if* $x(t)$ *satisfies the integral equation*

$$x(t) = \begin{cases} \mathcal{S}_{\lambda,\mu}(t)x_0 + \int_0^t \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau, x(\tau), Kx(\tau))d\tau, & t \in [0, t_1) \\ \mathcal{F}_k(t, x(t)), & t \in [t_k, s_k) \\ \mathcal{S}_{\lambda,\mu}(t - s_k)\mathcal{G}_k(s_k, x(s_k)) + \int_0^t \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau, x(\tau), Kx(\tau))dt, & \\ \qquad t \in [s_k, t_{k+1}) & \end{cases} \tag{8.3.2}$$

for all k.

The following theorem establishes the existence result for the Hilfer fractional integro-differential evolution system with classical condition (8.3.1).

Theorem 8.3.1. *If,*

(A1) The evolution operator $-A$ generates C_0 semigroup $S(t)$ for all $t \in [0, T_0]$.

(A2) The function $\mathcal{F} : [0, T_0] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is continuous with respect to t and there exist a positive constants f_1^* and f_2^* such that $||\mathcal{F}(t, u_1, v_1) - \mathcal{F}(t, u_2, v_2)|| \leq f_1^*||u_1 - u_2|| + f_2^*||v_1 - v_2||$ for $u_1, v_1, u_2, v_2 \in B_{r_0} = \{x \in \mathbb{X}; ||x|| \leq r_0\}$ for some r_0.

(A3) The operator $K : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ is continuous and there exist a constant k^* such that $||Ku - Kv|| \leq k^*||u - v||$ for $u, v \in B_{r_0}$.

(A4) The functions $\mathcal{G}_k : [t_k, s_k] \times \mathbb{X}$ are continuous and there exist a positive constants $0 < g_k^* < 1$ such that $||\mathcal{G}_k(t, u(t)) - \mathcal{G}_k(t, v(t))|| \leq g_k^*||u - v||$.

are satisfied, then the fractional integro-differential system (8.3.1) *with not-instantaneous impulses has a unique mild solution.*

Proof. Define the operator $\mathcal{P}$ on Banach space $\mathbb{X}$ by

$$\mathcal{P}x(t) = \begin{cases} \mathcal{P}_1 x(t), & t \in [0, t_1) \\ \mathcal{P}_{2k} x(t), & t \in [t_k, s_k) \\ \mathcal{P}_{3k} x(t), & t \in [s_k, t_{k+1}) \end{cases}$$

where, the operators $\mathcal{P}_1$, $\mathcal{P}_{2k}$ and $\mathcal{P}_{3k}$ are defined as

$$\begin{aligned}
\mathcal{P}_1 x(t) &= \mathcal{S}_{\lambda,\mu}(t)x_0 + \int_0^t \mathcal{K}_\lambda(t-\tau)\mathcal{F}(t, x(\tau), Kx(\tau))d\tau, \quad t \in [0, t_1) \\
\mathcal{P}_{2k} x(t) &= \mathcal{G}_k(t, x(t)), \quad t \in [t_k, s_k) \\
\mathcal{P}_{3k} x(t) &= \mathcal{S}_{\lambda,\mu}(t-s_k)\mathcal{G}_k(s_k, x(s_k)) + \int_0^t \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau, x(\tau), Kx(\tau))ds, \\
&\qquad\qquad t \in [s_k, t_{k+1})
\end{aligned}$$

for all $k = 1, 2, \cdots p$.

In view of this definition of the operator $\mathcal{P}$, the equation (8.3.2) has a unique solution if and only if the operator equation $x(t) = \mathcal{P}x(t)$ has a unique solution. This is possible if and only if each of $x(t) = \mathcal{P}_1 x(t)$, $x(t) = \mathcal{P}_{2k} x(t)$ and $x(t) = \mathcal{P}_{3k} x(t)$ has unique solution over the interval $[0, t_1)$, $[t_k, s_k)$ and $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$ respectively.
For all $t \in [0, t_1)$ and $u, v \in B_{r_0}$,

$$\begin{aligned}
||\mathcal{P}_1^{(n)} u(t) - \mathcal{P}_1^{(n)} v(t)|| \leq \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} & ||\mathcal{K}_\lambda(t-\tau_1)|| ||\mathcal{K}_\lambda(\tau_1-\tau_2)|| \cdots \\
& ||\mathcal{K}_\lambda(\tau_{n-1}-s)|| (f_1^* + k^* f_2^*)^n ||u-v|| \\
& ds d\tau_{n-1} \cdots d\tau_1
\end{aligned}$$

Assuming (A1), (A2) and (A3) and using lemma 8.2.2 and simplifying we get,

$$\begin{aligned}
||\mathcal{P}_1^{(n)} u(t) - \mathcal{P}_1^{(n)} v(t)|| &\leq \int_0^{t_1} \int_0^{t_1} \cdots \int_0^{t_1} \frac{t_1^{n(\lambda-1)} M^n (f_1^* + k^* f_2^*)^n}{(\Gamma(\lambda))^n} \\
&\quad \times ||u-v|| ds d\tau_{n-1} \cdots d\tau_1 \\
&\leq \frac{t_1^{n(\lambda-1)} M^n (f_1^* + k^* f_2^*)^n}{(\Gamma(\lambda))^n (n-1)!} \int_0^{t_1} (t_1-\tau)^{n-1} |u-v|| d\tau \\
&\leq \frac{t_1^{n\alpha} M^n (f_1^* + f_2^* k^*)^n}{n! (\Gamma(\alpha))^n} ||u-v|| \\
&\leq c^* ||u-v||.
\end{aligned}$$

Taking limit n tending to ∞ over interval $[0, t_1)$, $||\mathcal{P}_1^{(n)} u - \mathcal{P}_1^{(n)} v| \leq c^* ||u-v|| \to 0$ for fixed t_1. Therefore, there exist at least one m such that $\mathcal{P}_1^{(m)}$ is contraction on B_{r_0}. Thus, by general Banach contraction theorem the operator equation $x(t) = \mathcal{P}_1 x(t)$

has unique solution over the interval $[0, t_1)$.
For all $k = 1, 2, \cdots, p$, $t \in [t_k, s_k)$ and $u, v \in \mathbb{X}$ and assuming (A4)

$$||\mathcal{P}_{2k}u(t) - \mathcal{P}_{2k}v(t)|| = ||\mathcal{G}_k(t, u(t)) - \mathcal{G}_k(t, v(t))|| \leq g_k^*||u - v||.$$

Therefore, $\mathcal{P}_{2k}$ is contraction, and by Banach fixed point theorem the operator equation $x(t) = \mathcal{P}_{2k}x(t)$ has a unique solution for the interval $[t_k, s_k)$ for all $k = 1, 2, \cdots, p$. This means for all $k = 1, 2, \cdots, p$, $x(t) = \mathcal{G}_k(t, x(t))$ has unique solution for all $t \in [t_k, s_k)$. Using Lipschitz continuity of $\mathcal{G}_k$, the solution x is unique at s_k also.
For all $k = 1, 2, \cdots, p$, $t \in [s_k, t_{k+1})$ and $u, v \in B_{r_0}$,

$$\begin{aligned}||\mathcal{P}_{3k}^{(n)}u(t) - \mathcal{P}_{3k}^{(n)}v(t)|| \leq \int_{s_k}^{t}\int_{s_k}^{\tau_1}\cdots\int_{s_k}^{\tau_{n-1}} &||\mathcal{K}_\lambda(t-\tau_1)||||\mathcal{K}_\lambda(\tau_1-\tau_2)||\cdots \\ &||\mathcal{K}_\lambda(\tau_{n-1}-s)||(f_1^* + k^*f_2^*)^n ds d\tau_{n-1}\cdots d\tau_1\end{aligned}$$

Assuming (A1), (A2) and (A3) and using lemma 8.2.2 and simplifying we get,

$$\begin{aligned}||\mathcal{F}_{3k}^{(n)}u(t) - \mathcal{F}_{3k}^{(n)}v(t)|| \leq \int_{s_k}^{t_{k+1}}\int_{s_k}^{t_{k+1}}\cdots\int_{s_k}^{t_{k+1}} &\frac{(t_{k+1}-s_k)^{n(\lambda-1)}M^n(f_1^* + k^*f_2^*)^n}{(\Gamma(\lambda))^n} \\ &||u-v|| ds d\tau_{n-1}\cdots d\tau_1\end{aligned}$$

$$\begin{aligned}||\mathcal{P}_{3k}^{(n)}u(t) - \mathcal{P}_{3k}^{(n)}v(t)|| &\leq \frac{(t_{k+1}-s_k)^{n(\lambda-1)}M^n(f_1^* + f_2^*k^*)^n}{(n-1)!(\Gamma(\lambda))^n} \\ &\qquad \int_{s_k}^{t_{k+1}}(t_{k+1}-s)^{n-1}ds||u-v|| \\ &\leq \frac{(t_{k+1}-s_k)^{n\lambda}M^n(f_1^* + f_2^*k^*)^n}{n!(\Gamma(\lambda))^n}||u-v|| \\ &\leq c^*||u-v||.\end{aligned}$$

Over interval $[s_k, t_{k+1})$ and taking $n \to \infty$, $||\mathcal{P}_{3k}^{(n)}u - \mathcal{P}_{3k}^{(n)}v| \leq c^*||u - v|| \to 0$ for fixed sub-interval $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$. Thus, there exist at least one m such that $\mathcal{P}_{3k}^{(m)}$ is contraction on B_{r_0}. Thus by general Banach contraction theorem the operator equation $x(t) = \mathcal{P}_{3k}x(t)$ has unique solution over the interval $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$.
Hence, the operator equation $x(t) = \mathcal{P}(t)$ has a unique solution over the interval $[0, T_0]$ which is nothing but the mild solution of the equation (6.3.1). □

Example 8.3.1. *Consider the integro-differential Hilfer system of order* ($0 < \lambda \leq 1$ *and* $0 \leq \mu \leq 1$)*:*

$$\begin{aligned} D_t^{\lambda,\mu}u(t,\psi) &= u_{\psi,\psi}(t,\psi) + u(t,\psi)u_\psi(t,\psi) + \int_0^t e^{-u(\tau,\psi)}d\tau, \\ &\qquad\qquad t \in [0,1/3) \cup [2/3,1] \\ u(t,\psi) &= \frac{u(t,\psi)}{2(1+u^2(t,\psi))}, \ t \in [1/3,2/3) \end{aligned} \tag{8.3.3}$$

over the interval $[0,1]$ *with initial condition* $u(0,\psi) = u_0(\psi)$ *and boundary condition* $u(t,0) = u(t,1) = 0$*.*
The equation (8.3.3) *can be reformulated as fractional order abstract equation in* $\mathbb{X} = L^2([0,1],\mathbb{R})$ *as:*

$$\begin{aligned} D^{\lambda,\mu}z(t) &= -\mathcal{A}z(t) + \mathcal{F}(t,z(t),Kz(t)), \qquad t \in [0,1/3) \cup [2/3,1] \\ z(t) &= \mathcal{G}(t,z(t)) \qquad t \in [1/3,2/3) \end{aligned} \tag{8.3.4}$$

over the interval $[0,1]$ *by defining* $z(t) = u(t,\cdot)$*, operator* $-\mathcal{A}u = u''$ *(second order derivative with respect to* ψ*). The functions* $\mathcal{F}$ *and* $\mathcal{G}$ *over respected domains are as* $\mathcal{F}(t,z(t),Kz(t)) = (z^2(tt))'/2 + \int_0^t e^{-z(\tau)}d\tau$ *and* $\mathcal{G}(t,z(t)) = \frac{z(t)}{2(1+z^2(t))}$ *respectively.*

(1) The linear operator $\mathcal{A}$ *over the domain* $\mathcal{D}(\mathcal{A}) = \{u \in \mathbb{X}; u''$ *exist and continuous with* $u(0) = u(1) = 0\}$ *is self-adjoint, compact, and re-solvent. Therefore* A *is the infinitesimal generator of* C_0 *semi- group* $\mathcal{T}(t)$ *over the interval* $[0,1]$ *given by*

$$\mathcal{T}(t)u = \sum_{n=1}^{\infty} exp(-n^2\pi^2 t)\langle u,\phi_n\rangle\phi_n, \tag{8.3.5}$$

where $\phi_n(\psi) = \sqrt{2}sin(n\pi\psi)$ *for all* $n = 1,2,\cdots$ *is the orthogonal basis for the space* $\mathbb{X}$*.*

(2) The function $\mathcal{F} : [0,1] \times [0,1] \times \mathbb{X} \to \mathbb{X}$ *is continuous with respect to* t *and differentiable with respect to* z *for all* z *and hence* K *is Lipschitz continuous with respect to* z*. This means there exist positive constant* k^* *such that* $||K(t,z_1) - K(t,z_2)|| \leq k^*||z_1 - z_2||$*.*

(3) The function $\mathcal{F} : [0,1] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ *is continuous with respect to* t *and is differential with respect to argument* z *and* Kz*. Therefore there exist positive constants* f_1^* *and* f_2^* *such that* $||\mathcal{F}(t,z_1,Kz_1) - \mathcal{F}(t,z_2,Kz_2)|| \leq f_1^*||z_1 - z_2|| +$

$f_2^*||Kz_1 - Kz_2||$, $z_1, z_2 \in B_{r_0}$ *for some* r_0.

(4) The impulse $\mathcal{F}$ *is continuous with respect to* t *and Lipchitz continuous with respect to* z *with Lipschitz constant* $g^* = 1/2 < 1$.

Thus, by theorem-8.3.1 the equation (8.3.4) *has unique solution over* $[0, 1]$. *Hence, the equation* (8.3.3) *has a unique solution over the interval* $[0, 1]$.

8.4 Equation with Nonlocal Conditions

This section establishes the existence results for fractional order ($0 < \lambda \leq 1$ and $0 \leq \mu \leq 1$) non-instantaneous impulsive Hilfer integro-differential evolution systems with non-local initial conditions.

$$\begin{aligned} D_{0+}^{\lambda,\mu} x(t) &= -\mathcal{A}x(t) + \mathcal{F}\Big(t, x(t), \int_0^t a(t, \tau, x(\tau))d\tau\Big), \\ &\qquad\qquad t \in \Big[\cup [s_i, t_{i+1})\Big] \cup [s_p, T_0] \\ x(t) &= \mathcal{G}_k(t, x(t)),\ t \in [t_1, s_1) \cup [t_2, s_2) \cup \cdots [t_p, s_p) \\ x(0) &= x_0 + h(x) \end{aligned} \tag{8.4.1}$$

in the Banach space $\mathbb{X}$.

Definition 8.4.1. *The function* $x(t)$ *is called mild solution of the impulsive fractional equation* (8.4.1) *over the interval if* $x(t)$ *satisfies the integral equation*

$$x(t) = \begin{cases} \mathcal{S}_{\lambda,\mu}(t)[x_0 + h(x)] + \int_0^t \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau, x(\tau), Kx(\tau))d\tau,\ t \in [0, t_1) \\ \mathcal{G}_k(t, x(t)),\ t \in [t_k, s_k) \\ \mathcal{S}_{\lambda,\mu}(t - s_k)\mathcal{G}_k(s_k, x(s_k)) + \int_0^t \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau, x(\tau), Kx(\tau))dt, \\ \qquad\qquad t \in [s_k, t_{k+1}) \end{cases} \tag{8.4.2}$$

The following theorem establishes the existence of the solution for the non-local non-instantaneous fractional integro-differential evolution system (8.4.1) of Hilfer type over the interval $[0, T_0]$.

Theorem 8.4.1. *If,*

(B1) The evolution operator $-\mathcal{A}$ *generates* C_0 *semigroup* $S(t)$ *for all* $t \in [0, T_0]$.

(B2) The function $\mathcal{F}(t, \cdot, \cdot)$ *is continuous and* $\mathcal{F}(\cdot, u, v)$ *is measurable on* $[0, T] \times \mathbb{X} \times \mathbb{X}$. *Also, there exist* $\gamma \in (0, \lambda)$ *with* $m_f \in L^{\frac{1}{\gamma}}([0, T_0], \mathbb{R})$ *such that* $|\mathcal{F}(t, u, v)| \leq m_f(t)$ *for all* $u, v \in \mathbb{X}$.

(B3) The operator $K : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ *is continuous and there exist a constant* k^* *such that*
$||Ku - Kv|| \leq k^* ||u - v||$.

(B4) The operator $h : \mathbb{X} \to \mathbb{X}$ *is Lipschitz continuous with respect to* $u \in \mathbb{X}$ *with Lipschitz constant* $0 < h^* \leq 1$.

(B5) The functions $\mathcal{G}_k : [t_k, s_k] \times \mathbb{X}$ *are continuous and there exist a positive constants* $0 < g_k^* < 1$ *such that*
$||\mathcal{G}_k(t, u(t)) - \mathcal{G}_k(t, v(t))|| \leq g_k^* ||u - v||$.

are satisfied then the non-local non-instantaneous fractional order integro-differential evolution system (8.4.2) *has a mild solution provided* $M_0^* h^* < 1$ *and* $M_0^* g^* < 1$.

Proof. Using the lemma-9.2.2 and (B4),

$$|\mathcal{S}_{\lambda,\mu}(t)(x_0 + h(x))| \leq \frac{M(\mu(1-\lambda))t^{\lambda+\mu-\lambda\mu-1}}{\Gamma(\lambda+\mu-\lambda\mu-1)}(|x_0| + h^*||x|| + |h(0)|). \tag{8.4.3}$$

for all $x \in B_k = \{u \in \mathbb{X} : ||x|| \leq k\}$ for any positive constant k and $t \in [0, T_0]$. Using (B2), $(t-s)^{\lambda-1} \in L^{\frac{1}{1-\gamma}}[0, t)$ for all $t \in [0, T_0]$ and $\gamma \in (0, \lambda)$. Taking $M_1 = ||m_f||_{L^{\frac{1}{\gamma}}}$ and using Holder's inequality and assuming (B2), for $t \in [0, T_0]$

$$\begin{aligned} \int_0^t \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau, u(\tau), Ku(\tau))|ds &\leq \frac{M}{\Gamma(\lambda)}\left(\int_0^t (t-\tau)^{\frac{\lambda-1}{1-\gamma}} ds\right)^{1-\gamma} M_1 \\ &\leq \frac{MM_1(\gamma-1)}{\Gamma(\lambda)(\lambda-\gamma)} t^{\lambda-\gamma}. \end{aligned} \tag{8.4.4}$$

For $t \in [0, t_1)$ and for positive r consider F_1 and F_2 on B_r as,

$$\begin{aligned} F_1 x(t) &= \mathcal{S}_{\lambda,\mu}(t)(x_0 + h(x)) \\ F_2 x(t) &= \int_0^t \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau, x(\tau), Kx(\tau))d\tau \end{aligned}$$

The function $x(t)$ is a mild solution of the semi-linear fractional integro-differential equation if and only if the operator equation $x = F_1x + F_2x$ has a solution for $u \in B_r$ for some r. Therefore the existence of a mild solution of (8.4.1) over the interval $[0, t_1)$ is equivalent to determining a positive constant r_0, such that $F_1 + F_2$ has a fixed point on B_{r_0}.

Step:1 $||F_1u + F_2v|| \leq r_0$ for some positive r_0.

Let $u, v \in B_{r_0}$ where,

$$\begin{aligned} r_0 &= M_0^* \frac{|x_0| + |h(z)|}{1 - M_0^* h^*} + \frac{M_1^*}{(1 - M_0^* h^*)} t_1^{(\lambda-\gamma)}, \\ M_0^* &= \frac{M(\mu(1-\lambda)) t_1^{\lambda+\mu-\lambda\mu-1}}{\Gamma(\lambda+\mu-\lambda\mu-1)}, \end{aligned}$$

and

$$M_1^* = \frac{M M_1(\gamma - 1)}{\Gamma(\lambda)(\lambda - \mu)}$$

and considering

$$\begin{aligned} &|F_1u(t) + F_2v(t)| \\ &\quad \leq \left| \mathcal{S}_{\lambda,\mu}(t)(x_0 + h(u)) \right| + \left| \int_0^t \mathcal{K}_\lambda(t-\tau) \mathcal{F}(\tau, u(\tau), Ku(\tau)) d\tau \right| \\ &\quad \leq \frac{M(\mu(1-\lambda)) t^{\lambda+\mu-\lambda\mu-1}}{\Gamma(\lambda+\mu-\lambda\mu-1)} (|x_0| + h^*||u|| + |h(0)|) \\ &\qquad\qquad + \frac{M M_1(\gamma-1)}{\Gamma(\lambda)(\lambda-\gamma)} t^{\lambda-\gamma} \\ &\qquad\quad \text{(using inequalities (8.4.3) and (8.4.4))} \\ &\quad \leq r_0 \quad \text{(since, } M_0^* h^* < 1\text{).} \end{aligned}$$

Therefore, $||F_1u + F_2v|| \leq r_0$ for every pair $u, v \in B_{r_0}$.

Step: 2 To show F_1 is contraction on B_{r_0}, consider $u, v \in B_{r_0}$ and $t \in [0, t_1)$,

$$\begin{aligned} |F_1u(t) - F_1v(t)| &\leq \left| \mathcal{S}_{\lambda,\mu}(t)(x_0 + h(u)) - \mathcal{S}_{\lambda,\mu}(t)(x_0 + h(v)) \right| \\ &\leq \frac{M(\mu(1-\lambda)) t^{\lambda+\mu-\lambda\mu-1}}{\Gamma(\lambda+\mu-\lambda\mu-1)} h^* ||u - v|| \leq M_0^* h^* ||u - v|| \end{aligned}$$

and $M_0^* h^* < 1$ which implies F_1 is contraction.

Step: 3 To show F_2 is completely continuous operator on B_{r_0}, consider the sequence $\{u_n\}$ in B_{r_0} converging to $u \in B_{r_0}$ then,

$$\begin{aligned} &|F_2u_n(t) - F_2u(t)| \\ &\quad \leq \int_0^t |\mathcal{K}_\lambda(t-\tau)||\mathcal{F}(\tau, u_n(\tau), Ku_n(\tau))d\tau - \mathcal{F}(\tau, u(\tau), Ku(\tau))d\tau| \\ &\quad \leq C^* \int_0^t ||\mathcal{F}(\tau, u_n(\tau), Ku_n(\tau)) - \mathcal{F}(\tau, u(\tau), Ku(\tau))||d\tau, \end{aligned}$$

where, $c^* = \frac{M(\mu(1-\lambda))t^{\lambda+\mu-\lambda\mu-1}}{\Gamma(\lambda+\mu-\lambda\mu-1)}$ and using continuity of $\mathcal{F}$ with respect to the second and third argument $||F_2u_n - F_2u|| \to 0$ as $n \to \infty$. So, F_2 is continuous.
Now to show $\{F_2u, u \in B_{r_0}\}$ is relatively compact it is sufficient to show

(1) The family of functions $\{F_2u, u \in B_{r_0}\}$ is uniformly bounded and equicontinuous.
(2) For any $t \in [0, t_1)$, $\{F_2u(t), u \in B_{r_0}\}$ is relatively compact in $\mathbb{X}$.

Clearly, for any $u \in B_{r_0}$, $||F_2u|| \leq r_0$, this means that the family $\{F_2u, u \in B_{r_0}\}$ is uniformly bounded in $\mathbb{X}$.
For any $u \in B_{r_0}$ and $0 \leq t_1 < t_2 < t_1$,

$$\begin{aligned} &|F_2u(t_2) - F_2(t_1)| \\ &\quad = \left| \int_0^{t_2} \mathcal{K}_\lambda(t_2-\tau)\mathcal{F}(\tau, u(\tau), Ku(\tau))d\tau - \int_0^{t_1} \mathcal{K}_\lambda(t_1-\tau) \right. \\ &\qquad \left. \mathcal{F}(\tau, u(\tau), Ku(\tau))d\tau \right| \\ &\quad = \left| \int_{t_1}^{t_2} \mathcal{K}_\lambda(t_2-\tau)\mathcal{F}(\tau, u(\tau), Ku(\tau))d\tau + \int_0^{t_1} \mathcal{K}_\lambda(t_2-\tau) \right. \\ &\qquad \left. \mathcal{F}(\tau, u(\tau), Ku(\tau))d\tau - \int_0^{t_1} \mathcal{K}_\lambda(t_1-\tau)\mathcal{F}(\tau, u(\tau), Ku(\tau))d\tau \right| \\ &\quad \leq \left| \int_{t_1}^{t_2} \mathcal{K}_\lambda(t_2-\tau)\mathcal{F}(\tau, u(\tau), Ku(\tau))d\tau \right| + \left| \int_0^{t_1} \big[\mathcal{K}_\lambda(t_2-\tau) - \right. \\ &\qquad \left. \mathcal{K}_\lambda(t_1-\tau)\big]\mathcal{F}(\tau, u(\tau), Ku(\tau))d\tau \right| \\ &\quad \leq I_1 + I_2, \end{aligned}$$

where,

$$
\begin{aligned}
I_1 &= \left| \int_{t_1}^{t_2} \mathcal{K}_\lambda(t_2-\tau)\mathcal{F}(\tau,u(\tau),Ku(\tau))d\tau \right| \\
&\leq \frac{MM_1(\gamma-1)}{\Gamma(\lambda)(\lambda-\gamma)}(t_2-t_1)^{\lambda-\gamma} \\
&\qquad\qquad \text{(Applying inequality (8.4.4) over interval } [t_1,t_2]\text{)},
\end{aligned}
$$

this implies the integral $I_1 \to 0$ as $t_1 \to t_2$. Similarly,

$$
\begin{aligned}
I_2 &= \left| \int_0^{t_1} \big[\mathcal{K}_\lambda(t_2-\tau)-\mathcal{K}_\lambda(t_1-\tau)\big]\mathcal{F}(\tau,u(\tau),Ku(\tau))d\tau \right| \\
&= \Bigg| \int_0^{t_1} \big[(t_2-\tau)^{\lambda-1}\mathcal{Q}_\lambda(t_2-\tau)-(t_1-\tau)^{\lambda-1}\mathcal{Q}_\lambda(t_1-\tau)\big] \\
&\qquad\qquad \mathcal{F}(\tau,u(\tau),Ku(\tau))d\tau \Bigg| \\
&\leq \left| \int_0^{t_1} (t_2-\tau)^{\lambda-1}\big[\mathcal{Q}_\lambda(t_2-\tau)-\mathcal{Q}_\lambda(t_1-\tau)\big]\mathcal{F}(\tau,u(\tau),Ku(\tau))d\tau \right| \\
&\quad + \left| \int_0^{t_1} \big[(t_2-\tau)^{\lambda-1}-(t_1-\tau)^{\lambda-1}\big]\mathcal{Q}_\lambda(t_1-\tau)\mathcal{F}(\tau,u(\tau),Ku(\tau))d\tau \right|
\end{aligned}
$$

Assuming the (B1),(B2), (B3) and Holder inequality the integral

$$
\left| \int_0^{t_1} (t_2-\tau)^{\lambda-1}\big[\mathcal{Q}_\lambda(t_2-\tau)-\mathcal{Q}_\lambda(t_1-\tau)\big]\mathcal{F}(\tau,u(\tau),Ku(\tau))d\tau \right|
$$

and the integral

$$
\left| \int_0^{t_1} \big[(t_2-\tau)^{\lambda-1}-(t_1-\tau)^{\lambda-1}\big]\mathcal{Q}_\lambda(t_1-\tau)\mathcal{F}(\tau,u(\tau),Ku(\tau))d\tau \right|
$$

also vanishes as $t_1 \to t_2$. The vanishing of both the integral lead to the vanishing of I_2. Thus, $|F_2u(\tau_2) - F_2(\tau_1)|$ tends to zero as $t_1 \to t_2$ for independent choice of $u \in B_{r_0}$. Hence, the family $\{F_2u, u \in B_{r_0}\}$ is equicontinuous.

Define the family $X(t) = \{F_2u(t), u \in B_{r_0}\}$ for all $t \in [0,t_1)$. Clearly, $X(0)$ is relatively compact. Let, $t_0 \in [0,t_1)$ be fixed and for each $\epsilon \in [0,t_1)$, define an operator F_ϵ on B_{r_0} by formula

$$
F_\epsilon u(t) = \int_0^{t-\epsilon} \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau,u(\tau),Ku(\tau))d\tau.
$$

To show that the family $X(t)$ for all $t \in [0, t_1)$ is relatively compact consider,

$$\begin{aligned}
&|F_2u(t) - F_\epsilon u(t)| \\
&\quad = \Big| \int_0^t \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau, u(\tau), Ku(\tau))d\tau \\
&\qquad - \int_0^{t-\epsilon} \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau, u(\tau), Ku(\tau))d\tau \Big| \\
&\quad \le \int_\epsilon^t |\mathcal{K}_\lambda(t-\tau) f(\tau, u(\tau), Ku(\tau))d\tau| \\
&\quad \le \frac{MM_1(\gamma-1)}{\Gamma(\lambda)(\lambda-\gamma)}(t-\epsilon)^{\lambda-\gamma} \quad \text{(Applying inequality (8.4.4))}.
\end{aligned}$$

Thus, $X(t)$ is relatively compact, and hence, by Ascoli-Arzela theorem the operator F_2 is completely continuous on B_{r_0}. Using Krasnoselskii's fixed point theorem F_1+F_2 has a fixed point on B_{r_0} which is a mild solution of the equation (8.4.1) over the interval $[0, t_1)$.

On the interval $[t_k, s_k)$ for all $k = 1, 2, \cdots, p$ and fixed positive r_0 define the operators F_1 and F_2 on B_{r_0} as,

$$\begin{aligned}
F_1u(t) &= \mathcal{G}_k(t, u(t)) \\
F_2u(t) &= 0
\end{aligned}$$

assuming (B5) using Krasnoselskii's fixed point theorem, $u(t)$ is the mild solution of the non-instantaneous Hilfer integro-differential fractional evolution system over the interval $[t_k, s_k)$.

On the interval $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$ and for positive r we define F_1 and F_2 on B_r as,

$$\begin{aligned}
F_1x(t) &= \mathcal{S}_{\lambda,\mu}(t-s_k)\mathcal{G}_k(s_k, x(s_k)) \\
F_2x(t) &= \int_0^t \mathcal{K}_\lambda(t-\tau)\mathcal{F}(\tau, x(\tau), Kx(\tau))dt
\end{aligned}$$

then, the function $u(t)$ is the mild solution of Hilfer fractional integro-differential evolution system over the interval $[s_k, t_{k+1})$ if and only if the operator equation $x = F_1x + F_2x$ has a solution for $x \in B_r$ for some r. This is equivalent to the a mild solution of (8.4.1) over the interval $[s_k, t_{k+1})$.
Select,

$$r_0 = M_0^* \frac{|\mathcal{G}_k(\cdot, z)|}{1 - M_0^* g_k^*} + \frac{M_1^*}{(1 - M_0^* g_k^*)}(t_{k+1} - s_k)_1^{(\lambda-\gamma)},$$

$$M_0^* = \frac{M(\mu(1-\lambda))(t_{k+1}-s_k)_1^{\lambda+\mu-\lambda\mu-1}}{\Gamma(\lambda+\mu-\lambda\mu-1)},$$

, and

$$M_1^* = \frac{MM_1(\gamma-1)}{\Gamma(\lambda)(\lambda-\mu)}$$

and applying similar arguments as applied for interval $[0, t_1)$ and using Krasnoselskii's fixed point theorem $F_1 + F_2$ has a fixed point on B_{r_0} which is nothing but the mild solution of the Hilfer fractional integro-differential evolution system (8.4.1) over the interval $[s_k, t_{k+1})$. This completes the proof of the theorem. □

Example 8.4.1. *Consider, the Hilfer fractional partial integro-differential evolution system with nonlocal conditions:*

$$\begin{aligned} D_t^{1/2,1}u(t,\psi) &= u_{\psi,\psi}(t,\psi) + u(t,\psi)u_\psi(t,\psi) \\ &\quad + \frac{1}{50}\int_0^t e^{-u(\tau,\psi)}d\tau, \ t \in [0,1/3) \cup [2/3,1] \\ u(t,\psi) &= \frac{u(t,\psi)}{10(1+u^2(t,\psi))}, \ t \in [1/3,2/3). \end{aligned} \tag{8.4.5}$$

over the interval $[0,1]$ *with initial condition* $u(0,\psi) = u_0(\psi) + \sum_{i=1}^{2} \frac{1}{3^i}u(1/i,\psi)$ *and boundary condition* $u(t,0) = u(t,1) = 0$.
The equation (8.4.5) *can be reformulated as fractional order abstract equation in* $\mathbb{X} = L^2([0,1],\mathbb{R})$ *as:*

$$\begin{aligned} D^{1/2,1}z(t) &= -\mathcal{A}z(t) + \mathcal{F}(t,z(t),Kz(t)), \qquad & t \in [0,1/3) \cup [2/3,1] \\ z(t) &= \mathcal{G}(t,z(t)) & t \in [1/3,2/3) \end{aligned} \tag{8.4.6}$$

over the interval $[0,1]$ *by defining* $z(t) = u(t,\cdot)$, *operator* $-\mathcal{A}u = u''$ *(second order derivative with respect to* t*). The functions* f *and* g *over respected domains are defined as* $\mathcal{F}(t,z(t),Kz(t)) = \frac{1}{50}\int_0^t e^{-z(\tau)}d\tau$ *and* $g(t,z(t)) = \frac{z(t)}{10(1+z^2(t))}$ *respectively.*
The equation (8.4.6) *satisfies the conditions (B1-B5) of the hypothesis with* $M_0^*h^* < 1$ *and* $M_0^*g^* < 1$. *Hence the equation* (8.4.6) *has a mild solution over the interval* $[0,1]$.

8.5 Conclusion

This chapter established the results concerning the mild solutions of non-instantaneous impulsive fractional integro-differential evolution system on the Banach space $\mathbb{X}$ by considering classical as well as non-local conditions. These results are obtained using the concept of non-linear functional analysis and fixed point theorems. Using these results one can obtain a mild solution for the non-instantaneous impulsive Hilfer fractional integro-differential system.