Synopsis of the Thesis entitled

# CONTROLLABILITY ANALYSIS OF IMPULSIVE DYNAMICAL SYSTEMS USING FUNCTIONAL ANALYTIC APPROACH

Submitted to the

**The Maharaja Sayajirao University of Baroda**

In partial fulfilment of the requirements
for the award of Degree of

**Doctor of Philosophy**

in
**Applied Mathematics**

by

**Vishant Shah**

Research Guide

**Dr. Jaita Sharma**

Department of Applied Mathematics

Faculty of Technology & Engineering

The Maharaja Sayajirao University of Baroda.

# 1 Literature Survey

Numerous issues in the physical, chemical, and biological sciences that involve changes at the moment of time liken insertion or removal of biomass, populations of species that undergo abrupt changes, abrupt harvesting, seepage flow in porous media, anomalous diffusion, wave patterns, and transport are modelled using impulsive differential equations [1–15]. As a result, impulsive differential equations emerged as a key area of applied mathematics. Researchers have developed an interest in researching the qualitative characteristics, such as existence, uniqueness, and continuity of the classical as well as mild solutions, as a result of the numerous applications of impulsive differential equations. Numerous academics have looked at the asymptotic behaviour, existence, and uniqueness of impulsive differential equations of integer order using a variety of methodologies. These studies may be found in the journals that quote [22–27].The articles include the qualitative characteristics for the fractional order differential equations [28–33] and reference their in.

On the other hand, one of the important characteristics of the system that directs a given beginning state to a desired final state at a final time is controllability. controllability is to show the existence of a control function, which steers the solution of the system from its initial state to final state, where initial state and final state may vary over the entire space [34, 35]. Controllability of integer order impulsive systems are found in the articles [36–43]

# 2 Controllablity of Dynamical Systems

This section discussed the controllability of linear and nonlinear dinemical systems using functional analytic approach.

## 2.1 Linear System

Consider dynamical system with linear control,

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)u(t) \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

Here, for all $t$ the state $x0, x(t) \in \mathbb{R}^n$, $u \in L^2([t_0, t_1], \mathbb{R}^m)$, and $A(t), B(t)$ are matrices of dimensions $n \times n, n \times m$ respectively. Let $\Phi(t, t_0)$ be an transition matrix generated by the homogeneous system $x(t) = A(t)x(t)$, then the solution of (2.1) is given by:

$$x(t) = \Phi(t, t_0)x_0 + \int_0^t \Phi(t, s)B(s)u(s)ds \tag{2.2}$$

**Definition 2.1.** *The system* (2.1) *is controllable over the interval* $[0, t_1]$ *if for each pair of vectors* $x_0$ *and* $x_1$ *in* $\mathbb{R}^n$*, there is a control* $u \in L^2([t_0, t_1], \mathbb{R}^m)$ *such that the solution of* (2.2) *with* $x(0) = x_0$ *satisfies* $x(t_1) = x_1$.

**Theorem 2.1.** *The following statements are equivalent*

1 *The system* (2.1) *is controllable over the interval* $[0, t_1]$.

2 *The operator* $C : L^2([t_0, t_1], \mathbb{R}^m) \times \mathbb{R}^n$ *defined by* $Cu = \int_0^{t_1} \Phi(t_1, s)B(s)u(s)ds$*is onto.*

3 *The operator* $C* : \mathbb{R}^n \times L^2([t_0, t_1], \mathbb{R}^m)$ *(adjoint of* $C$*) is one one.*

4 *The controllability grammian* $W : \mathbb{R}^n \times \mathbb{R}^n$ *defined by* $W = (CC^*)$ *is non singular.*

5 *The control function* $u(t) = C^* W^{-1}(x_1 - \Phi(t_1, 0)x_0)$ *steers the system from initial state* $x_0$ *to desire final state* $x_1$ *at* $t = t_1$.

## 2.2 Nonlinear System

Consider a nonlinear system with linear control,

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)u(t) + f(t, x(t)) \\ x(0) &= x_0 \end{aligned} \tag{2.3}$$

Here, for all $t$ the state $x0, x(t) \in \mathbb{R}^n$, $u \in L^2([t_0, t_1], \mathbb{R}^m)$, $f : [0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n$ is nonlinear function, and $A(t), B(t)$ are matrices of dimensions $n \times n, n \times m$ respectively. Let $\Phi(t, t_0)$ be an transition matrix generated by the homogeneous system $x(t) = A(t)x(t)$, then the mild solution of (2.3) is given by:

$$x(t) = \Phi(t, t_0)x_0 + \int_0^t \Phi(t, s)B(s)u(s)ds + \int_0^t \Phi(t, s)f(s, x(s))ds \tag{2.4}$$

**Definition 2.2.** *The system* (2.3) *is controllable over the interval* $[0, t_1]$ *if for each pair of vectors* $x_0$ *and* $x_1$ *in* $\mathbb{R}^n$*, there is a control* $u \in L^2([t_0, t_1], \mathbb{R}^m)$ *such that*

$$x_1 = \Phi(t_1, t_0)x_0 + \int_0^{t_1} \Phi(t_1, s)B(s)u(s)ds + \int_0^{t_1} \Phi(t_1, s)f(s, x(s))ds. \tag{2.5}$$

In view of the equation (2.5) the problem of controllability of the nonlinear system (2.3) reduces to the solvability of the equation (2.5).

# 3 Research Work

Existence, uniqueness, controllability, and trajectory controllability of various impulsive systems of integer order as well as fractional order studied in this research work. The thesis consist of eight chapters.

## 3.1 Chapter:1

This chapter discusses the introduction, and mathematical preliminaries required for the research work.

## 3.2 Chapter:2

This chapter discusses the existence and uniqueness of classical and mild solution of generalized impulsive evolution equation

$$\begin{aligned} x'(t) &= Ax(t) + f_i(t, x(t), T_i x(t), S_i x(t)), \quad t \in [t_{i-1}, t_i), \quad t \neq t_i \\ x(0) &= x_0 \\ \Delta x(t_i) &= I_i x(t_i), t = t_i, \quad for \;\; i = 1, 2, 3, \ldots, N. \end{aligned} \tag{3.1}$$

over the finite interval $[0, T_0]$ in the Banach space $\mathbb{X}$. Here, $A : \mathbb{X} \to \mathbb{X}$ is the linear part of the evolution equation, for all $i$, $f_i : [0, T_0] \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ are nonlinear functions operated over the interval $[t_{i-1}, t_i)$, operators $T_i, S_i : \mathbb{X} \times \mathbb{X}$ are operators operated over the interval $[t_{i-1}, t_i)$, and $I_i : \mathbb{X} \to \mathbb{X}$ are the jumps at the time moments $t = t_i$. The result are obtained through operator semigroup, and Banach fixed point theorem.

## 3.3 Chapter:3

This chapter discusses the controllability of the system

$$\begin{aligned} x'(t) &= A(t)x(t) + f_k(t, x(t)) + B_k(t)u(t) \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \cdots, p \\ x(0) &= x_0 \\ \Delta x(t_k) &= E_k x(t_k) + F_k u(t_k) \quad t = t_k \end{aligned} \tag{3.2}$$

over the interval $[t_0, T]$. Here,for each $t \in [0, T_0]$ $x(t)$ is state of the system, $u(t)$ is controller of the system, for each $k = 1, 2, \cdots, p+1$ nonlinear function $f_k(\cdot, \cdot)$ : defined on the state space of the system and satisfies "Caratheodory condition", that is $f_k(\cdot, x)$ is measurable with respect to $t$ for all $x \in \mathbb{X}$ and $f_k(t, \cdot)$ is continuous with respect of $x$ for almost all $t \in [0, T_0]$. The results are obtained by transforming the controllability problem in to solvablility problem.

## 3.4 Chapter:4

This chapter discusses the trajectory controllability of

$$\begin{aligned} x'(t) &= Ax(t) + f(t, x(t)) + w(t) \quad t \in [s_k, t_k + 1), \quad for\ all\ k = 0, 1, 2 \cdots, p \\ x(t) &= g_k(t, x(t)) + w_k(t) \quad t \in [t_k, s_k) \end{aligned} \tag{3.3}$$

over the interval $[0, T]$ with classical conditions and non-local conditions in the Banach space $\mathbb{X}$. These results are obtained through the concept of nonlinear functional analysis.

## 3.5 Chapter:5

This chapter discusses the trajectory controllability of the system governed by a second-order evolution system

$$\begin{aligned} x'' &= Ax + f(t, x, x') + w(t) \\ x(0) &= x_{10}, \quad x'(0) = x_{20} \end{aligned} \tag{3.4}$$

over the finite interval $[0, T_0]$. Here, the state of the system belongs to the Banach space $\mathbb{X}$, $A$ is the linear operator defined on the Banach space, $f : [0, T_0] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is a nonlinear function, and $W(t)$ is the trajectory controller of the system.

## 3.6 Chapter:6

This chapter discusses the existence of mild solutions for the non-instantaneous generalized Caputo fractional differential equation

$$\begin{aligned} {}^cD^\lambda x(t) &= Ax(t) + f_k\left(t, x(t), \int_0^t a_k(s, t, x(t))ds\right), \ t \in [s_k, t_{k+1}), \ k = 1, 2, \cdots, p \\ x(t) &= g_k(k, x(t)), \ t \in [t_k, s_k) \end{aligned}$$

with local condition $x(0) = s_0$ and non-local condition $x(0) = x_0 + h(x)$ over the interval $[0, T_0]$ in a Banach space $\mathbb{X}$. Here $A : \mathcal{X} \to \mathbb{X}$ is linear operator, $P_k x = \int_0^t a_k(s, t, x(t))dt$ are nonlinear Volterra integral operator on $\mathbb{U}$, $f_k : [0, T_0] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ are nonlinear functions applied in the intervals $[s_k, t_{k+1})$ and $g_k : [0, T_0] \times \mathbb{X}$ are set of nonlinear functions applied in the interval $[t_k, s_k)$ for all $k = 1, 2, \cdots, p$.

### 3.7 Chapter:7

This chapter discusses the existence and uniqueness of classical and mild solutions for an instantaneous impulsive evolution equation

$$\begin{aligned} {}^cD^\alpha x(t) &= Ax(t) + f(t, x(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\ x(t_0) &= x_0 \end{aligned} \tag{3.5}$$

over the interval $[0, T_0]$ on a Banach space $\mathbb{X}$. Here, ${}^cD^\alpha$ denotes Caputo fractional differential operator of order $0 < \alpha \leq 1$, $A : \mathbb{X} \to \mathbb{X}$ is linear operator and $f : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ is nonlinear function. $I_k : \mathbb{X} \to \mathbb{X}$ are impulse operator at time $t = t_k$, fro $k = 1, 2, \cdots, p$. We also derived conditions in which the classical solution and mild solution of (3.5) coincide.

### 3.8 Chapter:8

This chapter discusses the trajectory controllability of infinite dimensional Hilfer fractional control systems of the form:

$$\begin{aligned} &\mathcal{D}_{0+}^{\lambda,\mu} x(t) + Ax(t) = g(t, x(t), \int_0^t a(t, s, x(s))ds) + w(t), \\ &\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)} x(0) = x_0, \end{aligned}$$

over the interval $[0, T_0]$ and non-local conditions $\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}[x(0) - h(x)] = x_0$ in the Banach space $\mathbb{X}$.

# 4 List of articles communicated and published in Journals

1 Existence and Uniqueness of Classical and Mild Solutions of Generalized Impulsive Evolution Equations, International Journal of Nonlinear Sciences and Numerical Soutions, 19(2018), 775-780.

2 Existence Results for the Fractional Order Generalized Cauchy Problem with Non-instataneous Impulses, Palestine Journal of Mathematics, Accepted for Publication.

3 Trajectory Controllability of the Systems Governed by Hilfer Fractional Systems, YMER, VOLUME 20 : ISSUE 11 (Nov) - 2021.

4 Trajectory Controllability of Dynamical Systems with Non-instantaneous Impulses, YMER, VOLUME 20 : ISSUE 11 (Nov) - 2021.

5 Existence and Uniqueness of Classical and Mild Solutions of Fractional Cauchy Problem with Impulses, Communicated

6 Controllability of Semilinear Generalized Impulsive Evolution System, To be communicated.

7 Trajectory controllability of second order impulsive systems, To be communicated.

# 5 Paper presented in Conferences

1 Existence and Uniqueness of Classical and Mild Solutions of Generalized Impulsive Evolution Equation in International Conference Discrete and Computational Mathematics (ICDCM-2017) orgenized by The Gandhigram Rural Institute-Deemed to be University, from 16-18 Feb, 2017.

2 Existence and Uniqueness of Classical and Mild Solutions of Impulsive Fractional Evolution Equation in 86th Annual Conference of Indian Mathematical Society (An international conference) organized by Vellore Institute of Technology, from 17-20 Dec, 2020.

# References

[1] D. D. Bainov and A. B. Dishliev, Population dynamics control in regard to minimizing the time necessary for the regeneration of a biomass taken away from the population, Appl.Math. Comput., 39(1990), 37-48.

[2] L. H. Erbe, H. I. Freedman, X. Z. Liu, and J. H. Wu, Comparison principles for impulsive parabolic equations with applications to models of single species growth, J. Austral. Math. Soc.32(1991), 382-400.

[3] M. Kirane and Yu. V. Rogovchenko Comparison results for systems of impulse parabolic equations with applications to population dynamics, Nonlinear Analysis, 28(1997), 263-277.

[4] E. Joelianto, Linear Impulsive differential equations for hybird system modelling, 2003 Europian Control Conference (EEC), pp. 3335-3340, Cambridge, UK, 1-4 Sept. 2003.

[5] K. Dishlieva, Impulsive Differential Equations and Applications, J Applied Computat Math, 2016(2016), 1-6.

[6] V. Milman, A. Myshkis, On the stability of motion in the presence of impulses, Siberian Math J, 1(1996), 233-237.

[7] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[8] Z. G. Li, Y. Wen, Y. C. Soh, Analysis and Design of Impulsive Control, IEEE Transaction on Automatic Control, 46(2001), 894-897.

[9] Yuri V. Rogovchenko, Nonlinear Impulse Evolution Systems and Applications to Population Models, J. Of Math. Anal and App.,207(1997), 300-315.

[10] I. Podlubny, Fractional Differential Equation, Academic Press, San Diego, 1999.

[11] M.Renardy, W. J. Hrusa, J. A. Nohel, Mathematical problems in viscoelasticity. Longman Scientific and technical, Newyork, (1987).

[12] J. H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Computer Methods in Applied Mechanics and Engineering. 167 (1998), 57-68.

[13] A. M. A. El-Sayed, Fractional order wave equation, International Journal of Theoretical Physics. 35(1996), 311-322.

[14] V. Gafiychuk, B. Datsan, V. Meleshko, Mathematical modeling of time fractional reaction-diffusion system, Journal of Computational and Applied Mathematics, 220 (2008), 215-225.

[15] R. Metzler, J. Klafter, The restaurant at the end of random walk, the recent developments in description of anomalous transport by fractional dynamics, Journal of Physics A: A Mathematical and General, 37(2004), 161-208.

[16] J. H. He, Some applications of nonlinear fractional differential equations and their approximations, Bulletin of Science and Technology, 15,(1999) 86-90.

[17] M. Tamsir, V. K. Srivastava, Revisiting the approximate analytical solution of fractional order gas dynamics equation, Alexandria Engineering Journal, 55 (2016), 867-874.

[18] O. S. Iyiola, G. Olay, The fractional Rosenau-Hyman model and its approximate solutioninka Ojo, Okpala Mmaduabuchi, Alexandria Engineering Journal, 55(2016), 1655-1659.

[19] J. C. Prajapati, K. B. Kachhia, S. P. Kosta, Fractional calculus approach to study temperature distribution within a spinning satellite, Alexandria Engineering Journal, 55(2016), 2345-2350.

[20] B. Bonilla, M. Rivero, L. Rodriguez-Germa, J. J. Trujillo, Fractional differential equations as alternative models to nonlinear differential equations, Applied Mathematics and Computation, 187(2007), 79–88.

[21] Y. Cheng, G. Guozhu, On the solution of nonlinear fractional order differential equations, Nonlinear Analysis: Theory, Methods and Applications, 310 (2005), 26-29.

[22] Y. Rogovchenko, Impusive evolution systems: main results and new trends, Dynamics Contin.Diser.Impulsive Sys., 3(1997), 57-88.

[23] J. H. Liu, Nonlinear impulsive evolution equations, Dynamics Contin.Diser.Impulsive System, 6(1999), 77-85.

[24] A. Anguraj and M. M. Arjunan, Existance and Uniqueness of Mild and Classical solutions of Impusive evolution equations, Ele. J. Of Diff. Eq., 111(2005), 1-8.

[25] P. Gonzalez and M. Pinto, Asymptotic behavior of impulsive differential equations, Rokey Mountain Journal of Mathematics, 26(1996), 165-173.

[26] M. U. Akhmet, On the general problem of stability for impulsive differential equations, J. Math. Anal. Appl, 288(2003), 182-196.

[27] V. Shah, R. K. George, J. Sharma, and P. Muthukumar, Existence and Uniqueness of Classical and Mild Solutions of Generalized Impulsive Evoution Equations, International Journal of Nonlinear Sciences and Numerical Soutions, 19(2018), 775-780.

[28] M. Benchohra, B. A. Slimani, Existence and uniqueness of solutions to impulsive fractional differential equations, Eel. J. Of Diff. Eq., 2009(2009), 1-11.

[29] G. Mophou, Existence and uniqueness of mild solutions to impulsive fractional differential equations. Nonlinear Analysis 72(2010),1604–1615.

[30] C. Ravichandran, M. Arjunan, Existence and Uniqueness results for impulsive fractional integro-differential equations in Banach spaces, International Journal of Nonlinear Science, 11(2011), 427-439.

[31] K. Balachandran,K. S. Kiruthika, J. J. Trujillo, Existence results for fractional impulsive integro-differential equations in Banach spaces, Commun Nonlinear Sci Numer Simulat. 16(2011), 1970-1977.

[32] H. R. Kataria and P. H. Patel, Congruency between classical and mild solutions of Caputo fractional impulsive evolution equation on Banach Space,International Journal of Advance Science and Technology, 29,3s(2020), 1923-1936.

[33] J. Borah and N. Bora, Existence of Mild Solution of A Class of Nonlocal Fracitonal Order Differential Equations with Non Instaneous Impulses, Fractional Calculus and Applied Analysis, 22(2019), 495-508.

[34] E. D. Sontag Deterministic Finite Dimensional Systems, Springer, (1998)

[35] R. W. Brockett, Finite Dimensional Linear Systems, John Wiley & Sons, USA (1970)

[36] S. Leela, F. A. McRae and S. Sivasundaram, Controllability of Impulsive Differential Equations, J. Math. Anal. App., 177(1993), 24-30.

[37] Z. Benzaid and M. Sznaier, Constrained Controllability of Linear Impulse Differential Systems, IEEE Trans. Auto. Control, 39(1994), 1064-1066.

[38] R. K. George, A. K. Nandakumaran and A. Arapostathis, A Note on Controllability of Impulsive Systems, J. Math. Anal. Appl., 241(2000), 276-283.

[39] Z.H. Guan, T.H. Qian, X. Yu, On controllability and observability for a class of impulsive systems, Systems and Control Letters, 47(2002), 247–257.

[40] G. Xie, L. Wang, Controllability and observability of a class of linear impulsive systems, J. Math. Anal. Appl., 304(2005), 336–355.

[41] S. Zhao, J. Sun, Controllability and observability for a class of time-varying impulsive systems, Nonlinear Analysis: Real World Applications, 10(2009), 1370–1380.

[42] R. K. George and J. P. Sharma, Exact controllability of Nonlinear Impulsive system, Proc. Dyna. Sys. App., 5(2005), 182-185.

[43] B. Dubey, R. K. George, On controllability of linear and semilinear impulsive systems, Numerical Functional Analysis and Optimization, 39(2018), 843-864.