Chapter 3

Controllability of Generalized Impulsive Systems

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This chapter discussed the exact controllability of the linear and nonlinear generalized impulsive evolution systems over the finite interval J_0. The exact controllability of the linear systems was achieved using the concept of operator semigroup, and the concepts of linear functional analysis, The exact controllability of semilinear systems using the concept of operator semigroup, nonlinear functional analysis, and generalized Banach fixed point theorem. To support the result, an application is included in this article.

3.1 Introduction

Many scientific and engineering processes have short-term perturbations in the state of any evolution system modeled in the impulsive differential equation or impulsive evolution system. Evolution systems have impulsive behavior often encountered in population dynamics [7, 124], fluctuation of oscillations due to external impulsive effect, the motion of pendulum under impulsive force [39], transfer of the orbit of satellite [151], and many other. For more details of applications, readers can refer to Laxmikantham et al [84].

The controllability of a system is one of its fundamental properties. It directs any system from a starting state to a desired ending state at a defined final time using the right function called controller of the system [22, 140]. The controllability of the time-invariant linear impulsive systems was investigated firstly by Leela et.al [85]. The controllability of linear impulsive systems with limitations was studied by Benzaid and Sznaier [19]. Using the controller in the final subinterval, George et al [53] explored the complete controllability of finite dimensional impulsive systems. Guan et al[57], Xie & Wang [153], and Zhao & Sun [159] explored controllability & observability for linear impulsive systems with no control constraints at the discontinuity. Using Lipschitzian criteria and Banach's fixed point theorem, George and Sharma [52] modified the system and developed certain adequate requirements for controllability for the semilinear impulsive system. Dubey and George [40] and analyze the complete controllability of finite dimensional linear and semilinear equations. Dubey and George [40] demonstrated that in impulsive systems "it is preferable to apply the control in early intervals rather than final intervals" and studied the entire controllability of finite dimensional linear and semilinear equations. Muni and George presented the controllability of time-varying linear systems on finite-dimensional Hilbert space [126].

In many impulsive systems, the perturbing forces may change after every impulse. To model these types of systems, a generalized impulsive system can be useful. The qualitative properties for the solutions of generalized evolution systems are found in Shah et al [135]. This motivates us to investigate the controllability results for the system:

$$
\begin{aligned}
x'(t) &= \mathcal{A}x(t) + \mathcal{F}_k(t, x(t), x(t)) + \mathcal{B}_k u(t) \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \cdots, \rho \\
x(0) &= x_0 \\
\Delta x(t_k) &= \mathcal{M}_k x(t_k) + \mathcal{N}_k u(t_k), \quad t = t_k, \quad k = 1, 2, \cdots, \rho
\end{aligned}
\tag{3.1.1}
$$

over the interval $[0, T_0]$, $0 < t_1 < t_2 \cdots, t_k, \cdots t_\rho < T_0$ are the impulse points. Here, the state $x(t)$ in the Hilbert space $\mathbb{X}$ for all $t \in J_0 = [0, T_0]$, $\mathcal{A}$, and $\mathcal{M}_k$ are linear operators on $\mathbb{X}$, $u \in L^2([0, T_0], \mathbb{U})$ $\mathcal{B}_k, \mathcal{N}_k : \mathbb{X} \times \mathbb{U}$ are bounded linear functions between Hilbert spaces $\mathbb{X}$ and $\mathbb{U}$, and $\mathcal{F}_k : [0, T_0] \times \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ are nonlinear functions. The system (3.1.1) has different perturbations after every impulse therefore, it is desirable to apply a controller at every subinterval after the impulses.

3.2 Controllability of Linear Impulsive Systems

To discuss the controllability of system (3.1.1), we develop the theory for the corresponding linear system of the form:

$$
\begin{aligned}
x'(t) &= \mathcal{A}x(t) + \mathcal{B}_k u(t) \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \cdots, \rho \\
x(0) &= x_0 \\
\Delta x(t_k) &= \mathcal{M}_k x(t_k) + \mathcal{N}_k u(t_k), \quad t = t_k, \quad k = 1, 2, \cdots, \rho
\end{aligned}
\tag{3.2.1}
$$

over the interval J_0.

Definition 3.2.1. *The mild solution for evolution system* (3.2.1) *is given by*

$$
x(t) = \mathcal{T}(t - t_{k-1})x(t_{k-1}^+) + \int_{t_{k-1}}^{t} \mathcal{T}(t - s)\mathcal{B}_k u(s)ds \tag{3.2.2}
$$

for all $t \in [t_{k-1}, t_k)$. Here, $\mathcal{T}(t)$ is the operator semi-group generated by linear operator $\mathcal{A}$ and $t_0 = 0$.

Definition 3.2.2. *(Exact Controllability)[86] The system* (3.2.1) *is exactly controllable across the range $[0, T_0]$ if, there exists a function $u \in L^2(J_0, \mathbb{X})$ called controller for every $x_0, x_1 \in L^2(J_0, \mathbb{X})$ such that the mild solution $x(t)$ of* (3.2.2) *corresponding to u satisfies $x(T_0) = x_1$.*

For discussion of the exact controllability of the evolution system (3.2.1) over subinterval $[t_{k-1}, t_k]$, introduce the operators $\mathcal{C}_k : L^2(J_0, \mathbb{U}) \to \mathbb{X},\ \forall k = 1, 2, \cdots, \rho + 1$by

$$\mathcal{C}_k u(t) = \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)\mathcal{B}_k u(s)ds, \tag{3.2.3}$$

whose adjoint operator $\mathcal{C}_k^* : \mathbb{X} \to L^2(J_0, \mathbb{U})$ is given by

$$\mathcal{C}_k^* z = \mathcal{B}_k^* \mathcal{T}^*(t_k - t)z. \tag{3.2.4}$$

Finally, define the operator $\mathcal{W}_k : L^2(J_0, \mathbb{X}) \to L^2(J_0, \mathbb{X})$ by

$$\mathcal{W}_k z = \mathcal{C}_k \mathcal{C}_k^* z = \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)\mathcal{B}_k \mathcal{B}^* T^*(t_k - s)zds. \tag{3.2.5}$$

Lemma 3.2.1. *The system*

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{B}_k u(t) \\ x(t_{k-1}) &= z_{k-1} \end{aligned} \tag{3.2.6}$$

is exactly controllable on the subinterval $[t_{k-1}, t_k]$ if, any one from below satisfied for some $\gamma > 0$, for all $x \in \mathbb{X}$.

(a) $Range(\mathcal{C}_k) = \mathbb{X}$.

(b) $||\mathcal{C}_k^* z||_{\mathbb{X}}^2 = \int_{t_{k-1}}^{t_k} ||(C_k^* z)(s)||_{\mathbb{U}}^2 ds \geq \gamma^2 ||z||_{\mathbb{X}}^2$.

(c) $\langle \mathcal{W}_k z, z \rangle \geq \gamma^2 ||z||_{\mathbb{X}}^2$.

(d) $\int_{t_{k-1}}^{t_k} ||\mathcal{B}_k^* \mathcal{T}^*(t_k - s)z||_{\mathbb{U}}^2 ds \geq \gamma^2 ||z||_{\mathbb{X}}^2$.

(e) $Ker(\mathcal{C}_k^*) = \{0\}$ and $Range(\mathcal{C}_k^*)$ is closed.

Proof of the theorem is in the same manner as discussed in the monograph of Curtain and Zwart [34].

Lemma 3.2.2. *The system (3.2.6) is exactly controllable on $[t_{k-1}, t_k]$ if and only if, the operator $\mathcal{W}_k$ is non-singular. Moreover the control $u \in L^2(J_0, \mathbb{U})$ steering an initial state z_{k-1} to the final state x_1 at time $t = t_k$ is given by*

$$u(t) = \mathcal{B}_k^* \mathcal{T}^*(t_k - t)\mathcal{W}_k^{-1}[x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1}] \tag{3.2.7}$$

Proof. The mild solution of the system over the interval $[t_{k-1}, t_k]$ is given by

$$x(t) = \mathcal{T}(t - t_{k-1}) + \int_{t_{k-1}}^{t} \mathcal{T}(t-s)\mathcal{B}_k u(s)ds \tag{3.2.8}$$

Suppose $\mathcal{W}_k$ is invertible therefore, plugging $u(t)$ from (3.2.7) in (3.2.8)

$$x(t) = \mathcal{T}(t - t_{k-1})z_{k-1} + \int_{t_{k-1}}^{t} T(t-s)\mathcal{B}_k\mathcal{B}_k^*\mathcal{T}^*(t_k - s)\mathcal{W}_k^{-1}[x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1}]ds$$

and at $t = t_k$

$$\begin{aligned} x(t_k) &= \mathcal{T}(t_k - t_{k-1})z_{k-1} + \int_{t_{k-1}}^{t_k} T(t_k-s)\mathcal{B}_k\mathcal{B}_k^*\mathcal{T}^*(t_k - s)\mathcal{W}_k^{-1}[x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1}]ds \\ &= \mathcal{T}(t_k - t_{k-1})z_{k-1} + \mathcal{W}_k\mathcal{W}_k^{-1}[x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1}] \\ &= x_1 \end{aligned}$$

Therefore, the evolution system steers to the desired final state x_1 at time $t = t_k$. Hence, the system (3.2.6) is exactly controllable on the subinterval $[t_{k-1}, t_k]$.

Conversely, suppose the system (3.2.6) is controllable. Therefore, by lemma-3.2.2 there exists $\gamma \in \mathbb{R}$ such that $\langle \mathcal{W}_k z, z\rangle \geq \gamma^2 ||z||_{\mathbb{X}}^2$ for $z \in \mathbb{X}$. Thus, $\mathcal{W}_k$ is injective.

Since, the system (3.2.6) is controllable therefore, $Range(C_k^*)$ is closed and for any $u \in L^2([t_{k-1}, t_k], \mathbb{U})$, $\mathcal{C}_k(u) = \mathcal{C}_k(u_1 + u_2)$ with $u_1 \in Range(\mathcal{C}_k^*)$ and $u_2 \in Nullity(\mathcal{C}_k^*) = \{0\}$. This implies, $\mathcal{C}_k = \mathcal{C}_k u_1 \in Range(\mathcal{C}_k\mathcal{C}_k^*)$. Thus, $Range(\mathcal{W}_k) = \mathbb{X}$. This leads to the surjectivity of the operator $\mathcal{W}_k$. Hence, $\mathcal{W}_k$ is invertible operator. This completes the proof. □

The next lemma is a consequence of the lemma-3.2.2.

Lemma 3.2.3. *If the evolution system* (3.2.6) *is exactly controllable on the subinterval* $[t_{k-1}, t_k]$ *then, the operator* $\mathcal{S}_k : \mathbb{X} \to L^2([t_{k-1}, t_k], \mathbb{U})$ *define by*

$$\begin{aligned} \mathcal{S}_k\zeta &= \mathcal{C}_k^*\mathcal{W}_k^{-1}\zeta \\ (\mathcal{S}_k\zeta)s &= \mathcal{B}_k^*\mathcal{T}^*(t-s)\mathcal{W}^{-1}\zeta \end{aligned} \tag{3.2.9}$$

is the right inverse of $\mathcal{C}_k$. *This means* $\mathcal{C}_k \circ \mathcal{S}_k = I$.

Assumptions 3.2.1. *We make the following assumption for discussion of the con-*

trollability of the system (3.2.1) *over the interval* $[0, T_0]$

(A1) Linear part of the system $\mathcal{A}$ generates C_0 semigroup of linear operator $\mathcal{T}(\cdot)$.

(A2) The operators $(I + \mathcal{M}_k)$ are non-singular for all k.

The following theorem discusses the exact controllability of the system (3.2.1).

Theorem 3.2.1. *Under the assumptions-3.2.1 the system is controllable over the interval $[0, T_0]$ if $\mathcal{W}_k$ are non-singular $\forall\ k = 1, 2, \cdots, \rho + 1$.*

Proof. On the subinterval $[t_0, t_1)$, the system becomes:

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{B}_1 u(t) \\ x(t_0) &= x_0, \end{aligned} \tag{3.2.10}$$

and the mild solution of the system (3.2.10) over this interval becomes:

$$x(t) = \mathcal{T}(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)\mathcal{B}_1 u(s)ds. \tag{3.2.11}$$

Applying assumptions-3.2.1, non-singularity of the operator $\mathcal{W}_1$, and using lemma-3.2.1 the evolution system (3.2.10) is exactly controllable over the subinterval $[t_0, t_1)$. The control $u(t)$ which steers the system (3.2.5) from initial state x_0 to x_1 at $t = t_1$ is given by

$$u(t) = \mathcal{B}_1^* T^*(t_1 - t)\mathcal{W}_1^{-1}[x_1 - \mathcal{T}(t_1 - t_0)x_0]. \tag{3.2.12}$$

Applying the control u, the state at $t = t_1$ becomes $x(t_1^-) = x_1$.

Over the subinterval $[t_1, t_2)$ system becomes:

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{B}_2 u(t) \\ x(t_1^+) &= (I + \mathcal{M}_1)x_1 + \mathcal{N}_1 u(t_1). \end{aligned}$$

To derive sufficient condition, assuming that $N_1 u(t_1) = 0$, the system becomes:

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{B}_2 u(t) \\ x(t_1^+) &= (I + \mathcal{M}_1)x_1, \end{aligned} \tag{3.2.13}$$

and the mild solution over this interval becomes:

$$x(t) = \mathcal{T}(t - t_1)(I + M_1)x_1 + \int_{t_1}^{t} T(t - s)\mathcal{B}_1 u(s)ds. \tag{3.2.14}$$

Assuming the hypotheses-3.2.1, non-singularity of $\mathcal{W}_2$, and using the lemma-3.2.2 the evolution system is controllable on the subinterval $[t_1, t_2)$. The controller $u(t)$ which steers state $(I + \mathcal{M}_1)x_1$ to the desired state x_1 at time-moment $t = t_2$ is

$$u(t) = \mathcal{B}_2^* \mathcal{T}^*(t_1 - t)\mathcal{W}_2^{-1}[x_1 - \mathcal{T}(t_1 - t_0)(I + \mathcal{M}_1)x_1]. \tag{3.2.15}$$

Continuing this process for all $k = 3, 4, \cdots, \rho$ and assuming $\mathcal{N}_k u(t_k) = 0$, the system over the subinterval $[t_{k-1}, t_k)$ becomes:

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{B}_k u(t) \\ x(t_{k-1}^+) &= (I + \mathcal{M}_{k-1})x_1. \end{aligned} \tag{3.2.16}$$

and mild solution of the system becomes:

$$x(t) = \mathcal{T}(t - t_{k-1})(I + \mathcal{M}_{k-1})x_1 + \int_{t_{k-1}}^{t} \mathcal{T}(t - t_{k-1})\mathcal{B}_k u(s)ds. \tag{3.2.17}$$

Assuming the hypotheses-3.2.1, non-singularity of $\mathcal{W}_k$, and using the lemma-3.2.2 the evolution system (3.2.16) is exactly controllable over the subinterval $[t_{k-1}, t_k)$. The controller $u(t)$ which steers the state $(I + \mathcal{M}_{k-1})x_1$ to the desired final state x_1 at time $t = t_k$ is

$$u(t) = \mathcal{B}_k^* \mathcal{T}^*(t_k - t)\mathcal{W}_k^{-1}[x_1 - \mathcal{T}(t_k - t_{k-1})(I + \mathcal{M}_k)x_1]. \tag{3.2.18}$$

Finally, on the subinterval $[t_\rho, T_0]$ considering $\mathcal{N}_\rho u(t_p) = 0$ evolution system becomes

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{B}_{\rho+1}u(t) \\ x(t_p^+) &= (I + \mathcal{M}_\rho)x_1, \end{aligned} \tag{3.2.19}$$

and the mild solution becomes:

$$x(t) = \mathcal{T}(t - t_\rho)(I + \mathcal{M}_\rho)x_1 + \int_{t_\rho}^{t} \mathcal{T}(t - s)\mathcal{B}_{\rho+1}u(s)ds. \tag{3.2.20}$$

Assuming the hypothesis-3.2.1, non-singularity of $\mathcal{W}_{\rho+1}$, and using the lemma=3.2.2

the evolution system (3.2.19) is exactly controllable over the interval $[t_p, T_0]$ and control $u(t)$ which steers the evolution system from the state $(I + \mathcal{M}_\rho)x_1$ to the desired final state x_1 at time $t = T_0$. Hence the proof. □

3.3 Controllability of Nonlinear Impulsive Systems

This section will discuss the exact controllability of the nonlinear generalized impulsive system (3.1.1) over the interval J_0.

Definition 3.3.1. *The system* (3.1.1) *is said to be exactly controllable over the interval J_0 if for all $x_0, x_1 \in \mathbb{X}$, there exist a control $u \in L^2(J_0, \mathbb{U})$ such that the corresponding mild solution $x(t)$ of* (3.1.1) *satisfies $x(T_0) = x_1$.*

Since the system is such that perturbing forces $\mathcal{F}_k$'s are changed after every time moment, therefore to control the system (3.1.1) over the entire interval J_0 one should apply controller for every subinterval $[t_{k-1}, t_k]$ for each $k = 1, 2, \cdots, \rho + 1$. For that first, consider the system

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}_k(t, x(t), u(t)) + \mathcal{B}_k u(t) \quad t \in [t_{k-1}, t_k) \\ x(t_{k-1}) &= z_{k-1}. \end{aligned} \tag{3.3.1}$$

Assuming $\mathcal{F}_k$'s are good enough so that the system (3.3.1) has the unique solution of the form

$$x(t) = \mathcal{T}(t - t_{k-1})z_{k-1} + \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)\mathcal{B}_k u(s)ds + \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)\mathcal{F}_k(s, x(s), u(s))ds \tag{3.3.2}$$

for all $u \in L^2([0, T_0], \mathbb{U})$.

Define the operator $\mathcal{G}_k : L^2([t_{k-1}, t_k], \mathbb{U}) \to \mathbb{X}$ by

$$\mathcal{G}_k u = \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)\mathcal{B}_k u(s)ds + \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)F_k(s, x(s), u(s))ds, \tag{3.3.3}$$

The following theorem gives an obvious characterization of the exact controllability of the system (3.3.1).

Theorem 3.3.1. *The system* (3.3.1) *is exactly controllable over the every subinterval* $[t_{k-1}, t_k]$ *if and only if Range*$(\mathcal{G}_k)$.

Now, assume that the corresponding linear system (3.2.5) is exactly controllable over subinterval $[t_{k-1}, t_k]$ therefore by lemma-3.2.3 there exist a steering operator $\mathcal{S}_k$ which is right inverse of $\mathcal{C}_k$.

Defining $\bar{\mathcal{G}}_k : \mathbb{X} \to \mathbb{X}$ by

$$\bar{\mathcal{G}}_k\zeta = (\mathcal{G}_k \circ \mathcal{S}_k)\zeta = \zeta + \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)\mathcal{F}_k(s, x_\zeta(s), (\mathcal{S}_k\zeta)(s))ds, \tag{3.3.4}$$

where $x_\zeta(\cdot)$ is the mild solution of the equation (3.3.2) corresponding to $u(t) = \mathcal{B}_k^*\mathcal{T}(t_k - t)\mathcal{W}_k^{-1}\zeta$. Hence, by defining the operator $H_k : \mathbb{X} \to \mathbb{X}$ by

$$H_k\zeta = \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)F_k(s, x_\zeta(s), (\mathcal{S}_k\zeta)(s)),$$

the equation (3.3.4) becomes

$$\bar{\mathcal{G}}_k\zeta = (I + H_k)\zeta. \tag{3.3.5}$$

Lemma 3.3.1. *If the operator* $H_k^{(N)}$ *on* $\mathbb{X}$ *for some* $N \geq 1$ *is contraction then* $(I + H_k)$ *is invertible.*

Proof. Since $H_k^{(N)}$ is a contraction for some $N \geq 1$ therefore by Generalized Banach fixed point theorem the equation $\zeta = -H_k\zeta$ has a unique solution on $\mathbb{X}$. This implies the invertibility of the operator $(I + H_k)$. □

The foregoing lemma discusses the exact controllability of the system (3.3.1) over the subinterval $[t_{k-1}, t_k]$ in the form of the abstract equation.

Theorem 3.3.2. *If the corresponding linear system of* (3.3.1) *is controllable over the interval* $[t_{k-1}, t_k]$ *and the nonlinear operator* $(I + H_k)$ *is invertible then nonlinear system* (3.3.1) *is exactly controllable over the interval* $[t_{k-1}, t_k]$ *and the controller* $u(t) = \mathcal{B}_k^*\mathcal{T}^*(t_k - s)\mathcal{W}_k^{-1}(I + H_k)^{-1}(x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1})$ *steers the given initial state* z_{k-1} *to desired state* x_1 *at time moment* $t = t_k$.

Proof. Plugging $u(t)$ in the (3.3.2) at $t = t_k$ and using nonsingularity $(I + H_k)$ the state of the system (3.3.1) becomes

$$\begin{aligned} x(t_k) &= \mathcal{T}(t_k - t_{k-1})z_{k-1} + \mathcal{G}_k\big[\mathcal{B}_k^*\mathcal{T}^*(t_k - s)\mathcal{W}_k^{-1}(I + H_k)^{-1}(x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1})\big] \\ &= \mathcal{T}(t_k - t_{k-1})z_{k-1} + \big[\mathcal{G}_k \circ \mathcal{S}_k\big] \circ \bar{\mathcal{G}_k}^{-1}\big(x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1}\big) \\ &= \mathcal{T}(t_k - t_{k-1})z_{k-1} + x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1} = x_1. \end{aligned}$$

Hence, the evolution system(3.3.1) is exactly controllable over the subinterval $[t_{k-1}, t_k]$. ☐

Considering the controller $u_0(t) = \mathcal{B}_k^*\mathcal{T}^*(t_k - s)\mathcal{W}_k^{-1}(I + H_k)^{-1}(x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1})$ and consider the following scheme

$$u_n(t) = \mathcal{B}_k^*\mathcal{T}^*(t_k - s)\mathcal{W}_k^{-1}(I + H_{k,n})^{-1}(x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1}) \tag{3.3.6}$$

$$\begin{aligned} x_{n+1}(t_k) = \mathcal{T}(t_k - t_{k-1})z_{k-1} + \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)\mathcal{B}_k u_n(s)ds \\ \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)\mathcal{F}_k(s, x_n(s), u_n(s))ds, \end{aligned} \tag{3.3.7}$$

where $H_{k,n}$ is defined as

$$H_{k,n}x_n = \int_{t_{k-1}}^{t_k} \mathcal{T}(t_k - s)\mathcal{F}_k(s, x_n(s), \big(\mathcal{S}_k x_n\big)((s))$$

The following assumptions are to be made to discuss the controllability of the system (3.3.1) over the subinterval $[t_{k-1}, t_k]$.

(B1) The operator semigroup generated by $\mathcal{A}$ is such that $||\mathcal{T}(t)|| \leq M$ for all $t \in J_0$, and $||\mathcal{B}_k|| \leq b_k^*$.

(B2) The perturbations f_k are measurable with respect to the first argument, and there exist constants f_{1k}^*, f_{2k}^* such that

$$||\mathcal{F}_k(t, x_1, u_1) - \mathcal{F}_k(t, x_2, u_2)|| \leq f_{1k}^*||x_1 - x_2|| + f_{2k}^*||u_1 - u_2||$$

for all $x_1, x_2 \in \mathbb{X}$, and $u_1, u_2 \in \mathbb{U}$.

Under assumptions (B1) and (B2),

$$\begin{aligned}||H_{k,n}^{(m)}x_1 - H_{k,n}^{(m)}x_2|| &\leq \frac{M^m(f_{1k}^* + b^*M||W_k||f_{2k}^*)^m}{(m-1)!}\int_{t_{k-1}}^{t_k}(t-s)^{n-1}ds||x_1 - x_2|| \\ &= \frac{M^m(f_{1k}^* + b^*M||W_k||f_{2k}^*)^m(t_k - t_{k-1})^m}{m!}||x_1 - x_2||.\end{aligned}$$

The value $c^* = \frac{M^m(f_{1k}^*+b^*M||W_k||f_{2k}^*)^m)(t_k-t_{k-1})^m}{m!} \to 0$ as $m \to \infty$. Therefore, there exists at least one N such that $H_{k,n}^{(N)}$ is contraction. The foregoing theorem discusses the controllability of a system (3.3.1) over the subinterval $[t_{k-1}, t_k]$.

Theorem 3.3.3. *If (B1) and (B2) are satisfied then, the evolution system* (3.3.1) *is exactly controllable on the subinterval* $[t_{k-1}, t_k]$ *with the controller* $u(t) = \mathcal{B}_k^*\mathcal{T}^*(t_k - s)\mathcal{W}_k^{-1}(I + H_k)^{-1}(x_1 - \mathcal{T}(t_k - t_{k-1})z_{k-1})$ *steer to the desired final state* x_1 *at* $t = t_k$.

The following theorem discusses the controllability of the nonlinear impulsive system (3.1.1).

Theorem 3.3.4. *If the corresponding linear system* (3.2.1) *is controllable and hypotheses (B1) and (B2) are satisfied then the nonlinear impulsive system is exactly controllable over the interval* J_0.

Proof. To discuss the exact controllability of system (3.1.1), a linear system is exactly controllable and also assumes that $\mathcal{N}_k u(t_k) = 0$ for all $k = 1, 2, \cdot, \rho$.
On the subinterval $[0, t_1)$, the evolution system becomes

$$\begin{aligned}x'(t) &= \mathcal{A}x(t) + \mathcal{F}_1(t, x(t), u(t)) + B_k u(t) \\ x(0) &= x_0.\end{aligned} \tag{3.3.8}$$

and using hypotheses (B1) and (B2) and applying the theorem-3.3.3 the evolution system is exactly controllable on the subinterval $[0, t_1)$ with control $u(t) = \mathcal{B}_1^*\mathcal{T}^*(t_1 - s)\mathcal{W}_1^{-1}(I + H_1)^{-1}(x_1 - \mathcal{T}(t_1)x_0)$ and the state of the system at $t = t_1$ steers to x_1 at $t = t_1$.

Over the subinterval $[t_1, t_2)$, the evolution system becomes

$$\begin{aligned}x'(t) &= \mathcal{A}x(t) + \mathcal{F}_2(t, x(t), u(t)) + B_k u(t) \\ x(t_1) &= (I + \mathcal{M}_1)x_1.\end{aligned} \tag{3.3.9}$$

and using hypotheses (B1) and (B2), applying the theorem- 3.3.3 the system is exactly controllable over with controller $u(t) = \mathcal{B}_2^*\mathcal{T}^*(t_2 - s)\mathcal{W}_2^{-1}(I + H_2)^{-1}(x_1 - \mathcal{T}(t_2 - t_1)(I + \mathcal{M}_1)x_1)$ the interval $[t_1, t_2)$ and state of the system steer at x_1 at $t = t_2$.

Continuing this process up to final interval $[t_\rho, T_0]$ the system becomes

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}_{\rho+1}(t, x(t), u(t)) + \mathcal{B}_{\rho+1}u(t) \\ x(t_\rho) &= (I + \mathcal{M}_\rho)x_1. \end{aligned} \tag{3.3.10}$$

and assuming hypotheses (B1) and (B2), and applying the theorem-3.3.3 the evolution system (3.3.10) is exactly controllable on the subinterval $[t_\rho, T_0]$. The controller $u(t) = \mathcal{B}_{\rho+1}^*\mathcal{T}^*(T_0 - s)\mathcal{W}_{\rho+1}^{-1}(I + H_{\rho+1})^{-1}(x_1 - \mathcal{T}(T_0 - t_{\rho-1})(I + \mathcal{M}_\rho)x_1)$ steers the state of system (3.3.10) to x_1 at time $t = T_0$.
This completes the proof of the theorem. □

3.4 Application

To apply the derived results, consider the nonlinear impulsive Korteweg–De Vries (KdV) equation. This is a mathematical model of waves on shallow water surfaces with obstacles at fixed moments of time represented by

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} &= \mathcal{B}_1 u(x,t) + F_1(t, x, w, u) \quad t \in [0, t_1) \\ \frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} &= \mathcal{B}_2 u(x,t) + F_2(t, x, w, u) \quad t \in [t_1, T_0] \\ \Delta w(x, t_1) &= 2w(x, t_1) + 3u(x, t_1) \end{aligned} \tag{3.4.1}$$

with periodic boundary conditions

$$\frac{\partial^i w}{\partial x^i}(0, t) = \frac{\partial^i w}{\partial x^i}(2\pi, t), \quad \forall i = 0, 1, 2 \tag{3.4.2}$$

and initial condition

$$w(x, 0) = w_0(x). \tag{3.4.3}$$

Here, u is the control function, and the linear operators $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined by

$$\mathcal{B}_1 u(x,t) = g_1(x)\left[u(x,t) - \int_0^{2\pi} g_1(\zeta)u(\zeta,t)d\zeta\right] \tag{3.4.4}$$

$$\mathcal{B}_2 u(x,t) = g_2(x)\left[u(x,t) - \int_0^{2\pi} g_2(\zeta)u(\zeta,t)d\zeta\right] \tag{3.4.5}$$

where, g_1 and g_2 are continuous functions over the interval $[0, 2\pi]$.

Russell [125] discussed the exact controllability of the linear KdV without impulses, and George et. al. [25] included the nonlinear perturbation of the KdV and discussed the exact controllability without impulses. This section discusses the exact controllability of the impulsive system (3.4.1).

Fix $\mathbb{X} = L^2([0, 2\pi], \mathbb{R})$ and defined an operator $\mathcal{A}$ having domain

$$\mathcal{D}(\mathcal{A}) = \left\{\zeta \in \mathbb{H}^{(2)}[0, 2\pi]; \zeta^i(0) = \zeta^i(2\pi)\right\} \text{ by } \mathcal{A}w = -\frac{\partial^3 w}{\partial x^3}.$$

From the Lemma-5.2 in Chapter-8 of Pazi [110], the operator $\mathcal{A}$ generates C_0 semigroup $\mathcal{T}(\cdot)$ satisfying such $||\mathcal{T}(t)|| \leq M$ for some $M \geq 0$, and for all $t \in [0, T_0]$.

Taking $X(t) = w(\cdot, t)$, and $U(t) = u(\cdot, t)$ system transform into abstract system of the form

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{B}_k U(t) + F_k(t, x(t), u(t)), \quad t \in [0, t_1) \cup [t_1, T_0) \\ \Delta x(t_1) &= 2x(t_1) + 3u(t_1) \end{aligned} \tag{3.4.6}$$

over the Space $\mathbb{X}$. Since, the operator A generates C_0 operator semigroup and the jump $\mathcal{M}_1 x(t_1) = 2x(t_1)$ is such that $(I + \mathcal{M}_1)$ is invertible and $\mathcal{B}_1, \mathcal{B}_2$ are bounded. Thus the corresponding linear system

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{B}_k u(t), \quad t \in [0, t_1) \cup [t_1, T_0] \\ \Delta x(t_1) &= 2x(t_1) + 3u(t_1) \end{aligned}$$

is exactly controllable and the controller defined by

$$u(t) = \begin{cases} \mathcal{B}_1^* \mathcal{T}^*(t_1 - t)\mathcal{W}_1^{-1}[x_1 - \mathcal{T}(t_1)x_0] & t \in [0, t_1) \\ \mathcal{B}_2^* \mathcal{T}^*(T_0 - t)\mathcal{W}_2^{-1}[x_1 - \mathcal{T}(t_1)(I + \mathcal{M}_1)x_1] & t \in (t_1, T_0] \end{cases} \tag{3.4.7}$$

Also, if

(1) $\mathcal{F}_1$, and F_2 are measurable with respect to argument t.

(2) $\mathcal{F}_i$'s are continuous with respect to X and U also there exist constants F_i^* satisfy $||\mathcal{F}_i(t, X_1, U_1) - \mathcal{F}_i(t, X_2, U_2)|| \leq F_i^* \big(||X_1 - X_2|| + ||U_1 - U_2||\big)$

then the evolution system (3.4.1) is exactly controllable over the entire interval $[0, T_0]$.

3.5 Conclusion

There are various ways to discuss the controllability of the nonlinear impulsive system one of the ways is to observe the system up to the final impulse moment and then apply the control only over the final interval $[t_\rho, T_0]$. But applying the a controller in this way one needs to apply a huge amount of potential in a very short interval of time and due to this system may become unstable. In this article, we have tried to apply the controller in every subinterval which will give more stability to the system.