Chapter 5

Convergence and integrability of rational and double rational trigonometric series

Complex numbers and sequences of bounded variation are crucial tools in analysing the convergence of trigonometric series, simplifying calculations and offering elegant representations. They provide deeper insights into function behaviour, making them essential in mathematical analysis. Bounded variation sequences play a crucial role in studying functions' convergence, continuity, and differentiability, particularly in the context of Fourier analysis. In the Fourier series, these sequences are vital for approximating functions, ensuring accuracy in representations, and facilitating various operations. Additionally, sequences of bounded variation of higher order further contribute to the analysis and approximation of functions, especially those with oscillatory behaviour.

In 1954, Ul'yanov [64] obtained significant results for sine and cosine series where $f(x) = \sum_{n=1}^{\infty} a_n \cos nx$ and $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$ were considered. It was shown that if $\{a_n\}_{n=1}^{\infty}$ is a null sequence of bounded variation then $f, g \in L^p[0, 2\pi)$ for any $0 < p < 1$. In 1984, Stanojevic [58] considered the complex trigonometric series with its coefficients being a null sequence of generalized bounded variation, specifically bounded variation of order $m,\ m \geq 1$ and obtained the results related to convergence and integrability of such trigonometric series. Later, in 2004, Kaur et al. [32] extended these results for double trigonometric series. It is worth noting that if the coefficients of the trigonometric series are Fourier coefficients of some functions, then the trigonometric series becomes the Fourier series of that function.

We have derived inspiration to investigate rational trigonometric series with coefficients of bounded variation sequences, recognizing their significance in the broader context of mathematical analysis. The importance of bounded variation sequences in studying convergence and integrability of functions, as highlighted in the preceding text, has motivated our interest in exploring their application to rational trigonometric series. This endeavour aims to enhance our understanding of the behaviour of functions and contribute to advancing mathematical analysis.

The main focus of this chapter is to study analogous results by replacing trigonometric series with rational trigonometric series. Here, we are considering the following rational orthogonal system $\{\phi_n(e^{ix})\}_{n=-\infty}^{\infty}$, where

$$\phi_0(e^{ix}) = 1,\ \phi_n(e^{ix}) = \frac{\sqrt{1-r^2}e^{ix}}{1-re^{ix}} \left(\frac{e^{ix}-r}{1-re^{ix}}\right)^{n-1},\ \phi_{-n}(e^{ix}) = \overline{\phi_n(e^{ix})}, \quad (5.1)$$

$n \in \mathbb{N},\ x \in \mathbb{T}$ and $r \in [0,1)$. Clearly, (5.1) is obtained from the rational orthogonal system by putting $\alpha_k = r,\ \forall n \in \mathbb{N}$. Note that if $r = 0$, then the above rational orthogonal system $\{\phi_n(e^{ix})\}_{n=-\infty}^{\infty}$ reduces to the exponential system $\{e^{inx}\}_{n=-\infty}^{\infty}$. The following inequality can be easily deduced,

$$|\phi_n(e^{ix})| \leq \frac{1+r}{1-r},\ \forall n \in \mathbb{Z}. \quad (5.2)$$

5.1 Rational trigonometric series

The rational trigonometric series is defined as

$$\sum_{n=-\infty}^{\infty} c(n)\ \phi_n(e^{ix}), \quad (5.3)$$

where $\{c(n)\}_{n=1}^{\infty}$ is a sequence of complex numbers and $x \in \mathbb{T} := [0, 2\pi)$.

Theorem 5.1.1. *If for some $m \in \mathbb{N}$, a complex sequence $\{c(n)\}_{n\in\mathbb{Z}} \in \mathcal{BV}^m$ then the rational trigonometric series* (5.3)

(i) *converges pointwise to some function* $f(x)$ *for every* $x \in \mathbb{T} \setminus \{0\}$.
(ii) *converges in* $L^p(\mathbb{T})$*-metric to* f *for any* $0 < p < \frac{1}{m}$.

Proof. Part (i): Let $\omega := \omega(x) = 1 - \frac{1-re^{ix}}{e^{ix}-r}$, where $x \in \mathbb{T} \setminus \{0\}$.
Note that for $n \in \mathbb{Z} \setminus \{0, 1\}$ and $x \in \mathbb{T} \setminus \{0\}$,

$$\omega(x)\phi_n(e^{ix}) = \phi_n(e^{ix}) - \phi_{n-1}(e^{ix}). \tag{5.4}$$

Also, we have

$$\frac{\sin\left(\frac{x}{2}\right)}{1+r} \leq \frac{|\omega(x)|}{2} \leq 1, \;\; x \in \mathbb{T}. \tag{5.5}$$

Let N be positive integer such that $N > 1$ and $Q = \{n \in \mathbb{Z} \setminus \{0, 1\} : -N \leq n \leq N\}$. Also, let partial sum of series (5.3) be $S_N(x) := \sum_{n=-N}^{N} c(n)\phi_n(e^{ix})$. Thus, by (5.4), we get

$$\begin{aligned}
\omega^m \sum_{n \in Q} c(n)\phi_n(e^{ix}) & \\
&= \sum_{n=-N}^{-1} \Delta^m c(n)\phi_n(e^{ix}) + \sum_{t=0}^{m-1} \omega^{m-1-t}\Delta^t c(0)\phi_{-1}(e^{ix}) \\
&- \sum_{t=0}^{m-1} \omega^{m-1-t}\Delta^t c(-N)\phi_{-N-1}(e^{ix}) + \sum_{n=2}^{N} \Delta^m c(n)\phi_n(e^{ix}) \\
&+ \sum_{t=0}^{m-1} \omega^{m-1-t}\Delta^t c(N+1)\phi_N(e^{ix}) - \sum_{t=0}^{m-1} \omega^{m-1-t}\Delta^t c(2)\phi_1(e^{ix}).
\end{aligned} \tag{5.6}$$

Therefore, for $x \in \mathbb{T} \setminus \{0\}$,

$$\begin{aligned}
S_N(x) = c(0) + c(1)\phi_1(e^{ix}) & \\
+ \frac{1}{\omega^m} \Bigg(& \sum_{n=-N}^{-1} \Delta^m c(n)\phi_n(e^{ix}) + \sum_{t=0}^{m-1} \omega^{m-1-t}\Delta^t c(0)\phi_{-1}(e^{ix}) \\
- & \sum_{t=0}^{m-1} \omega^{m-1-t}\Delta^t c(-N)\phi_{-N-1}(e^{ix}) + \sum_{n=2}^{N} \Delta^m c(n)\phi_n(e^{ix}) \\
+ & \sum_{t=0}^{m-1} \omega^{m-1-t}\Delta^t c(N+1)\phi_N(e^{ix}) - \sum_{t=0}^{m-1} \omega^{m-1-t}\Delta^t c(2)\phi_1(e^{ix}) \Bigg).
\end{aligned} \tag{5.7}$$

Now, in view of (5.2),(5.5),(5.7) and by definition of $\mathcal{BV}^m$, we get that the rational orthogonal series (5.3) converges pointwise to some $f(x)$ for all $x \in \mathbb{T} \setminus \{0\}$.

Part (ii): For $x \in \mathbb{T} \setminus \{0\}$ and a positive integer N such that $N > 1$, we have,

$$f(x) - S_N(x) = \frac{1}{\omega^m}\left(\sum_{|n| \geq N+1} \Delta^m c(n)\phi_n(e^{ix}) + \sum_{t=0}^{m-1} \omega^{m-1-t}\Delta^t c(-N)\phi_{-N-1}(e^{ix}) \right.$$
$$\left. - \sum_{t=0}^{m-1} \omega^{m-1-t}\Delta^t c(N+1)\phi_N(e^{ix})\right).$$

Let $0 < p < \frac{1}{m}$. Then, by (5.2) and (5.5), we get

$$|f(x) - S_N(x)|^p \leq \left(\frac{1+r}{(1-r)|\omega|^m}\right)^p \left(\sum_{|n| \geq N+1} |\Delta^m c(n)| + 2^m \sum_{t=0}^{m-1} |\Delta^t c(-N)| \right.$$
$$\left. + 2^m \sum_{t=0}^{m-1} |\Delta^t c(N+1)|\right)^p.$$

Since $mp < 1$, by (5.5)

$$\int_{\mathbb{T}} \frac{dx}{|\omega|^{mp}} \leq \left(\frac{1+r}{2}\right)^{mp} \int_{\mathbb{T}} \frac{dx}{\sin^{mp}(\frac{x}{2})} \leq C_{rmp},$$

where C_{rmp} is some absolute constant.
Thus by definition of $\mathcal{BV}^m$, we get

$$\left(\int_{\mathbb{T}} |f(x) - S_N(x)|^p\right)^{1/p} \to 0 \text{ as } N \to \infty.$$

Hence, we get the result. □

Remark 17. If $r = 0$ in Theorem 5.1.1, then we get the analogous result for classical trigonometric series obtained by Stanojevic [58], as the rational orthogonal system $\{\phi_n(e^{ix})\}_{n=-\infty}^{\infty}$ reduces to the exponential system $\{e^{inx}\}_{n=-\infty}^{\infty}$. Similarly, for $r = 0$ and $m = 1$, we get the analogous result by Ul'yanov[64].

5.2 Double rational trigonometric series

The double rational trigonometric series is defined as

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c(j,k)\phi_j(e^{ix})\phi_k(e^{iy}), \tag{5.8}$$

where $\{c(j,k) : -\infty < j,k < \infty\}$ is a double sequence of complex numbers and $(x,y) \in \mathbb{T}^2$.

The rectangular partial sums of double rational trigonometric series (5.8) are given by

$$S_{JK}(x,y) = \sum_{|j|\le J} \sum_{|k|\le K} c(j,k)\phi_j(e^{ix})\phi_k(e^{iy}). \tag{5.9}$$

Theorem 5.2.1. *If for some $m \in \mathbb{N}$, a double complex sequence $\{c(j,k)\}_{(j,k)\in\mathbb{Z}^2} \in \mathcal{BV}_2^m$, then the double rational trigonometric series* (5.8)

(i) *converges regularly to some function $f(x,y)$ for every $(x,y) \in (\mathbb{T}\setminus\{0\})^2$.*

(ii) *converges in $L^p(\mathbb{T}^2)$-metric to f for any $0 < p < \frac{1}{m}$ when* $\min(j,k) \to \infty$.

Proof. Part (i): Let $\omega(x) = 1 - \frac{1-re^{ix}}{e^{ix}-r}$ where $x \in \mathbb{T}\setminus\{0\}$, J and K be positive integers greater than 1, $R = \{j \in \mathbb{Z}\setminus\{0,1\} : -J \le j \le J\}$ and $Q = \{k \in \mathbb{Z}\setminus\{0,1\} : -K \le k \le K\}$. Also, let $x, y \in \mathbb{T}\setminus\{0\}$.

Thus, by (5.4) and in view of (5.6) and (cf. [32, Lemma 2.1]), for $a \in \{0,1\}$,we get

$$\begin{aligned}
&\omega(x)^m \sum_{j\in R} c(j,a)\phi_j(e^{ix}) \\
&\quad= \sum_{j\in R} \Delta_{m0}c(j,a)\phi_j(e^{ix}) + \sum_{t=0}^{m-1} \omega(x)^{m-1-t}[\Delta_{t0}c(0,a)\phi_{-1}(e^{ix}) \\
&\qquad - \Delta_{t0}c(-J,a)\phi_{-J-1}(e^{ix}) + \Delta_{t0}c(J+1,a)\phi_J(e^{ix}) - \Delta_{t0}c(2,a)\phi_1(e^{ix})] \\
&\quad:= A_{j,a},
\end{aligned}$$

$$\omega(y)^m \sum_{k\in Q} c(a,k)\phi_k(e^{iy})$$

$$
\begin{aligned}
&= \sum_{k\in Q} \Delta_{0m} c(a,k)\phi_k(e^{iy}) + \sum_{t=0}^{m-1} \omega(y)^{m-1-t}[\Delta_{0t} c(a,0)\phi_{-1}(e^{iy}) \\
&\quad - \Delta_{0t} c(a-K)\phi_{-K-1}(e^{iy}) + \Delta_{0t} c(a,K+1)\phi_K(e^{iy}) - \Delta_{0t} c(a,2)\phi_1(e^{iy})] \\
&:= B_{a,k},
\end{aligned}
$$

and

$$
\begin{aligned}
&\omega(x)^m \omega(y)^m \sum_{j\in R}\sum_{k\in Q} c(j,k)\phi_j(e^{ix})\phi_k(e^{iy}) \\
&= \sum_{j\in R}\sum_{k\in Q} \Delta_{mm} c(j,k)\phi_j(e^{ix})\phi_k(e^{iy}) \\
&+ \sum_{j\in R}\sum_{t=0}^{m-1} \omega(y)^{m-1-t}\left[\Delta_{mt} c(j,K+1)\phi_j(e^{ix})\phi_K(e^{iy}) + \Delta_{mt} c(j,0)\phi_j(e^{ix})\phi_{-1}(e^{iy})\right] \\
&- \sum_{j\in R}\sum_{t=0}^{m-1} \omega(y)^{m-1-t}\left[\Delta_{mt} c(j,2)\phi_j(e^{ix})\phi_1(e^{iy}) + \Delta_{mt} c(j,-K)\phi_j(e^{ix})\phi_{-K-1}(e^{iy})\right] \\
&+ \sum_{k\in Q}\sum_{s=0}^{m-1} \omega(x)^{m-1-s}\left[\Delta_{sm} c(J+1,k)\phi_J(e^{ix})\phi_k(e^{iy}) + \Delta_{sm} c(0,k)\phi_{-1}(e^{ix})\phi_k(e^{iy})\right] \\
&- \sum_{k\in Q}\sum_{s=0}^{m-1} \omega(x)^{m-1-s}\left[\Delta_{sm} c(2,k)\phi_1(e^{ix})\phi_k(e^{iy}) + \Delta_{sm} c(-J,k)\phi_{-J-1}(e^{ix})\phi_k(e^{iy})\right] \\
&+ \sum_{s=0}^{m-1}\sum_{t=0}^{m-1} \omega(x)^{m-1-s}\omega(y)^{m-1-t}\left[\Delta_{st} c(J+1,K+1)\phi_J(e^{ix})\phi_K(e^{iy})\right. \\
&\quad + \Delta_{st} c(J+1,0)\phi_J(e^{ix})\phi_{-1}(e^{iy}) + \Delta_{st} c(0,K+1)\phi_{-1}(e^{ix})\phi_K(e^{iy}) \\
&\quad + \Delta_{st} c(0,0)\phi_{-1}(e^{ix})\phi_{-1}(e^{iy}) - \Delta_{st} c(2,K+1)\phi_1(e^{ix})\phi_K(e^{iy}) \\
&\quad - \Delta_{st} c(2,0)\phi_1(e^{ix})\phi_{-1}(e^{iy}) - \Delta_{st} c(-J,K+1)\phi_{-J-1}(e^{ix})\phi_K(e^{iy}) \\
&\quad - \Delta_{st} c(-J,0)\phi_{-J-1}(e^{ix})\phi_{-1}(e^{iy}) - \Delta_{st} c(J+1,2)\phi_J(e^{ix})\phi_1(e^{iy}) \\
&\quad - \Delta_{st} c(J+1,-K)\phi_J(e^{ix})\phi_{-K-1}(e^{iy}) - \Delta_{st} c(0,2)\phi_{-1}(e^{ix})\phi_1(e^{iy}) \\
&\quad - \Delta_{st} c(0,-K)\phi_{-1}(e^{ix})\phi_{-K-1}(e^{iy}) + \Delta_{st} c(2,2)\phi_1(e^{ix})\phi_1(e^{iy}) \\
&\quad + \Delta_{st} c(2,-K)\phi_1(e^{ix})\phi_{-K-1}(e^{iy}) + \Delta_{st} c(-J,2)\phi_{-J-1}(e^{ix})\phi_1(e^{iy}) \\
&\quad \left. + \Delta_{st} c(-J,-K)\phi_{-J-1}(e^{ix})\phi_{-K-1}(e^{iy})\right] \\
&:= C_{j,k}.
\end{aligned}
$$

Now,

$$\begin{aligned}S_{JK}(x,y) =& c(0,0)+c(0,1)\phi_1(e^{iy})+c(1,0)\phi_1(e^{ix})+c(1,1)\phi_1(e^{ix})\phi_1(e^{iy})\\ &+\frac{A_{j,0}+\phi_1(e^{iy})A_{j,1}}{\omega(x)^m}+\frac{B_{0,k}+\phi_1(e^{ix})B_{1,k}}{\omega(y)^m}+\frac{C_{j,k}}{\omega(x)^m\omega(y)^m}. \quad (5.10)\end{aligned}$$

Note that, in view of definition of $\mathcal{BV}_2^m$, we get

$$\sum_{j\in R}\sum_{t=0}^{m-1}|\Delta_{mt}c(j,K+1)|\le C_m\sup_{k\in U}\sum_{|j|\le J}|\Delta_{m0}c(j,k)|\to 0 \text{ as } \min(J,K)\to\infty,$$

$$\sum_{s=0}^{m-1}\sum_{t=0}^{m-1}|\Delta_{st}c(J+1,K+1)|\le C_m\sup_{|j|\ge J+1,\ |k|\ge K+1}|c(j,k)|\to 0 \text{ as } \min(J,K)\to\infty,$$

and

$$\sum_{s=0}^{m-1}\sum_{t=0}^{m-1}|\Delta_{st}c(0,0)|\le C_m;$$

where $U=\{k\in\mathbb{Z}: K+1\le k\le K+m\}$ and C_m is an absolute constant not necessarily the same at each occurrence. Similarly, we can solve for other sums of $C_{j,k}$ and (5.10).

Thus, in view of (5.2),(5.5),(5.10), in the view of [12, Theorem 2.1] and by definition of $\mathcal{BV}_2^m$, we get that the double rational orthogonal series (5.8) converges $f(x,y)$ as $\min(j,k)\to\infty$ for all $(x,y)\in(\mathbb{T}\setminus\{0\})^2$. Also for $(x,y)\in(\mathbb{T}\setminus\{0\})^2$, by following similar steps as in Theorem 5.1.1, in the view of [12, Theorem 2.1] and by (5.2), the row series $\sum_{j=-\infty}^{\infty}c(j,k)\phi_j(e^{ix})\phi_k(e^{iy})$ converges for each fixed value of k and the column series $\sum_{k=-\infty}^{\infty}c(j,k)\phi_j(e^{ix})\phi_k(e^{iy})$ converges for each fixed value of j. Thus, we get that the double rational orthogonal series (5.8) converges regularly to $f(x,y)$ for all $(x,y)\in(\mathbb{T}\setminus\{0\})^2$.

Part (ii): For $x,y\in\mathbb{T}\setminus\{0\}$ and positive integers J and K such that $J,K>1$, we have,

$$f(x,y)-S_{JK}(x,y)$$

$$= \left[-\sum_{t=0}^{m-1} \omega(x)^{m-1-t} \{ -\Delta_{t0} c(-J,0) \phi_{-J-1}(e^{ix}) + \Delta_{t0} c(J+1,0) \phi_J(e^{ix}) \} \right.$$

$$\left. + \sum_{|j| \geq J+1} \Delta_{m0} c(j,0) \phi_j(e^{ix}) \right] \frac{1}{\omega(x)^m} + \frac{\phi_1(e^{iy})}{\omega(x)^m} \left[\sum_{|j| \geq J+1} \Delta_{m0} c(j,1) \phi_j(e^{ix}) \right.$$

$$\left. - \sum_{t=0}^{m-1} \omega(x)^{m-1-t} \{ -\Delta_{t0} c(-J,1) \phi_{-J-1}(e^{ix}) + \Delta_{t0} c(J+1,1) \phi_J(e^{ix}) \} \right]$$

$$+ \left[-\sum_{t=0}^{m-1} \omega(y)^{m-1-t} \{ -\Delta_{0t} c(0,-K) \phi_{-K-1}(e^{iy}) + \Delta_{0t} c(0,K+1) \phi_K(e^{iy}) \} \right.$$

$$\left. + \sum_{|k| \geq K+1} \Delta_{0m} c(0,k) \phi_k(e^{iy}) \right] \frac{1}{\omega(y)^m} + \frac{\phi_1(e^{ix})}{\omega(y)^m} \left[\sum_{|k| \geq K+1} \Delta_{0m} c(1,k) \phi_k(e^{iy}) \right.$$

$$\left. - \sum_{t=0}^{m-1} \omega(y)^{m-1-t} \{ -\Delta_{0t} c(1,-K) \phi_{-K-1}(e^{iy}) + \Delta_{0t} c(1,K+1) \phi_K(e^{iy}) \} \right]$$

$$+ \frac{1}{\omega(x)^m \omega(y)^m} \left[\sum_{|j| \geq J+1} \sum_{|k| \geq K+1} \Delta_{mm} c(j,k) \phi_j(e^{ix}) \phi_k(e^{iy}) \right.$$

$$- \sum_{|j| \geq J+1} \sum_{t=0}^{m-1} \omega(y)^{m-1-t} \Delta_{mt} c(j,K+1) \phi_j(e^{ix}) \phi_K(e^{iy})$$

$$- \sum_{|j| \geq J+1} \sum_{t=0}^{m-1} \omega(y)^{m-1-t} \Delta_{mt} c(j,0) \phi_j(e^{ix}) \phi_{-1}(e^{iy})$$

$$+ \sum_{|j| \geq J+1} \sum_{t=0}^{m-1} \omega(y)^{m-1-t} \Delta_{mt} c(j,2) \phi_j(e^{ix}) \phi_1(e^{iy})$$

$$+ \sum_{|j| \geq J+1} \sum_{t=0}^{m-1} \omega(y)^{m-1-t} \Delta_{mt} c(j,-K) \phi_j(e^{ix}) \phi_{-K-1}(e^{iy})$$

$$- \sum_{|k| \geq K+1} \sum_{s=0}^{m-1} \omega(x)^{m-1-s} \Delta_{sm} c(J+1,k) \phi_J(e^{ix}) \phi_k(e^{iy})$$

$$- \sum_{|k| \geq K+1} \sum_{s=0}^{m-1} \omega(x)^{m-1-s} \Delta_{sm} c(0,k) \phi_{-1}(e^{ix}) \phi_k(e^{iy})$$

$$+ \sum_{|k| \geq K+1} \sum_{s=0}^{m-1} \omega(x)^{m-1-s} \Delta_{sm} c(2,k) \phi_1(e^{ix}) \phi_k(e^{iy})$$

$$
\begin{aligned}
&+ \sum_{|k|\geq K+1} \sum_{s=0}^{m-1} \omega(x)^{m-1-s} \Delta_{sm} c(-J,k) \phi_{-J-1}(e^{ix}) \phi_k(e^{iy}) \\
&- \sum_{s=0}^{m-1} \sum_{t=0}^{m-1} \omega(x)^{m-1-s} \omega(y)^{m-1-t} \big[\Delta_{st} c(J+1,K+1) \phi_J(e^{ix}) \phi_K(e^{iy}) \\
&\quad + \Delta_{st} c(0,K+1) \phi_{-1}(e^{ix}) \phi_K(e^{iy}) - \Delta_{st} c(2,K+1) \phi_1(e^{ix}) \phi_K(e^{iy}) \\
&\quad - \Delta_{st} c(-J,K+1) \phi_{-J-1}(e^{ix}) \phi_K(e^{iy}) - \Delta_{st} c(-J,0) \phi_{-J-1}(e^{ix}) \phi_{-1}(e^{iy}) \\
&\quad - \Delta_{st} c(J+1,2) \phi_J(e^{ix}) \phi_1(e^{iy}) - \Delta_{st} c(J+1,-K) \phi_J(e^{ix}) \phi_{-K-1}(e^{iy}) \\
&\quad - \Delta_{st} c(0,-K) \phi_{-1}(e^{ix}) \phi_{-K-1}(e^{iy}) + \Delta_{st} c(2,-K) \phi_1(e^{ix}) \phi_{-K-1}(e^{iy}) \\
&\quad + \Delta_{st} c(-J,2) \phi_{-J-1}(e^{ix}) \phi_1(e^{iy}) + \Delta_{st} c(-J,-K) \phi_{-J-1}(e^{ix}) \phi_{-K-1}(e^{iy}) \\
&\quad + \Delta_{st} c(J+1,0) \phi_J(e^{ix}) \phi_{-1}(e^{iy}) \big] \Bigg].
\end{aligned}
$$

Let $0 < p < \frac{1}{m}$. Hence, $mp < 1$ and therefore by (5.5)

$$
\int\int_{\mathbb{T}^2} \frac{dx\ dy}{|\omega(x)\omega(y)|^{mp}} \leq \left(\frac{1+r}{2}\right)^{2mp} \int\int_{\mathbb{T}^2} \frac{dx\ dy}{\sin^{mp}(\frac{x}{2}) \sin^{mp}(\frac{y}{2})} \leq C_{rmp},
$$

where C_{rmp} is some absolute constant.
Thus by (5.2), (5.5), and definition of $\mathcal{BV}_2^m$, we get

$$
\left(\int\int_{\mathbb{T}^2} |f(x,y) - S_{JK}(x,y)|^p\right)^{1/p} \to 0 \text{ as } J,K \to \infty.
$$

Hence, we get the result. □

Remark 18. Note that if we take $r = 0$ in Theorem 5.2.1, we get results for classical double trigonometric series. Results for double trigonometric series for bounded variation of some order m and order 1 can be found in [32] and [40], respectively.