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The classical Fourier series is generalized to various orthogonal Fourier series to provide flexibility in approximating different types of functions, better convergence property, numerical efficiency and applications in different areas of Mathematics, Engineering and Physics. The Fourier series, the Legendre Fourier series, the Chebychev Fourier series and the Walsh Fourier series are suitable for approximating smooth periodic functions with no singularities [16], smooth bounded functions [10], analytic functions [29] and binary functions [13] respectively. Depending on the type of function to be approximated, different types of orthogonal Fourier series provide different convergence properties. The Fourier series, the Legendre Fourier series, the Chebychev Fourier series and the Walsh Fourier series converge uniformly for smooth functions , converge uniformly for square integrable functions, converge uniformly for analytic functions and converge for periodic continuous functions respectively. There are many more such orthogonal Fourier series. The choice of orthogonal Fourier series depends on the specific problem or application and the required accuracy. The classical Fourier series cannot be used to approximate non-periodic functions and functions with discontinuities or singularities, to overcome such limitations rational Fourier series is more suitable [12, 9]. Moreover, with the appropriate selection of parameters, the rational Fourier series exhibits faster convergence and better accuracy than the classical Fourier series for certain functions [22, 23]. It should be noted that the computation of rational Fourier series can be more complex and computationally intensive than Fourier series. Besides theoretical applications, the rational Fourier series finds numerous other applications in fields such as control theory [7], system identification [2], signal compression [19], denoising [36], and more.

In the 1920s, the rational orthogonal system was independently defined by Malmquist [20] and Takenaka [26]. Thus, the rational orthogonal system is also referred to as Takenaka-Mamquist (or Malmquist-Takenaka) orthogonal system. The rational orthogonal system is defined as follows

$$\phi_0(e^{ix}) = 1, \phi_n(e^{ix}) = \frac{\sqrt{1-|\alpha_n|^2}e^{ix}}{1-\overline{\alpha_n}e^{ix}} \prod_{k=1}^{n-1} \frac{e^{ix}-\alpha_k}{1-\overline{\alpha_k}e^{ix}}, \phi_{-n}(e^{ix}) = \overline{\phi_n(e^{ix})}, \ \forall n \in \mathbb{N}. \tag{1}$$

Here, $\{\alpha_n\}_{n\in\mathbb{N}}$ is complex sequence such that α_k's are in open unit disk $\mathbb{D}$. Achieser [1] observed that the system in (1) is complete in $L^2[0, 2\pi]$ if and only if $\sum_{n=1}^{\infty}(1-|\alpha_k|) = \infty$. The simplest way to satisfy the previous completeness condition is to assume

$$\sup|\alpha_k| := r < 1. \tag{2}$$

In 1956, Džrbašyan [11] worked on the concept of rational Fourier series with the orthogonal system as a rational orthogonal system. He obtained some properties of rational Dirichlet Kernel and proved the Jordan theorem and the Dini-Lipschitz test for the rational Fourier series under the assumption of condition (2).

Definition 1. If f is 2π periodic integerable function, then the rational Fourier series of f is defined as

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\phi_n(e^{ix}), \tag{3}$$

where $\hat{f}(n)$ is the n^{th} rational Fourier coefficient of f, given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\overline{\mathbb{T}}} f(x)\overline{\phi_n(e^{ix})}dx.$$

If $\alpha_k = 0$ in (1), $\forall k \in \mathbb{N}$, then series in (3) becomes Fourier series of f. Let $\alpha_k = |\alpha_k|e^{ix_k}$. Then, $\phi_n(e^{ix}) = \rho_n(x)e^{i\theta_n(x)}$ (cf. [11]), where

$$\rho_n(x) = \sqrt{\frac{1-|\alpha_n|^2}{1-2|\alpha_n|\cos(x-x_n)+|\alpha_n|^2}}$$

and

$$\theta_n^{'}(x) = \sum_{k=1}^{n-1}\frac{1-|\alpha_k|^2}{1-2|\alpha_k|\cos(x-x_k)+|\alpha_k|^2} + \frac{1}{2}\left(\frac{1-|\alpha_n|^2}{1-2|\alpha_n|\cos(x-x_n)+|\alpha_n|^2}\right) + \frac{1}{2}.$$

Here, $\theta_n(x)$ is a differentiable and strictly increasing function on $[0, 2\pi]$ [6, cf. p.465].

In the sequel, we assume that
i) Condition (2) holds,
ii) $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}, \mathbb{Z}^{*2} = \mathbb{Z}^*\times\mathbb{Z}^*$,...,$\mathbb{Z}^{*N} = \mathbb{Z}^*\times ... \times\mathbb{Z}^*$(N times),
iii) $\mathbb{T} = [0, 2\pi), \overline{\mathbb{T}}^N = \overline{\mathbb{T}}\times ... \times\overline{\mathbb{T}}$ (N times),
iv) $\{\varphi(n)\}_{n=1}^{\infty}$ is a real sequence such that $\varphi(1) \geq 2$ and $\varphi(n) \uparrow \infty$ as $n \to \infty$.

Note that, there are certain properties of the rational Fourier series which differs from the classical Fourier series as you can see from the example below that one of the basic result related to convolution fails for rational Fourier coefficients.

Definition 2. [40] If f and g are two integral functions on $\overline{\mathbb{T}}$ then convolution product of f and g, denoted by $f * g$ is given by

$$f * g(x) = \frac{1}{2\pi}\int\int_{\overline{\mathbb{T}}^2} f(t)g(x-t)dt;\ \forall x \in \overline{\mathbb{T}}.$$

If $c_n(f)$ and $c_n(g)$ are Fourier coefficients of 2π periodic integrable functions f and g then it is clear that

$$c_n(f * g) = c_n(f)\ c_n(g).$$

But, the above inequality does not hold for rational Fourier coefficients as $f(x) = g(x) = \sin x,\ n = 1$ and $\alpha_1 = \frac{1}{2}$ gives $\hat{f}(1) = \hat{g}(1) = \frac{-\sqrt{3}i}{4}$ and $\widehat{f * g}(1) = \frac{-\sqrt{3}}{8}$. Thus, it is interesting to note the difference in properties between rational Fourier series and classical Fourier series.

In 1881, Jordan [15] introduced the concept of bounded variation. The pursuit of elegance and/or generality by mathematicians in addressing specific problems has resulted in fascinating extensions of the concept of bounded variation. This has led to the emergence of novel categories of functions with generalized bounded variations. Some of the generalizations of Jordan's bounded variation class are due to Wiener [39], Waterman [37], Kita and Yoneda [18], Akhobadze [3, 4], and Vyas [32]. Properties of the Fourier series of functions with these generalized variations, such as the order of

Fourier coefficients, the convergence of Fourier series, and estimates of partial sums of Fourier series, have been proved for classical Fourier series and further extended for double or multiple Fourier series. The concept of generalized bounded variation has also been developed for sequences and is useful for results related to trigonometric series [25] and is extended for double trigonometric series [17]. In the thesis, we derive analogous results for the rational Fourier series and rational trigonometric series for one and several variables.

The thesis consists of five chapters. The main objective of Chapter 1. is to introduce the subject matter of the thesis by exploring some developments in the field.

Chapter 2. In this chapter, we present the results related to the order of magnitude of the rational Fourier coefficients of functions with generalized bounded variations, such as $\Lambda BV(p(n)\uparrow p,\varphi,\overline{\mathbb{T}})$ and $B\Lambda(p(n)\uparrow\infty,\varphi,\overline{\mathbb{T}})$.

Definition 3. [32, Definition 1.1] Let f be a complex valued measurable function defined on $I:=[a,b]$; $\Lambda=\{\lambda_k\}_{k=1}^{\infty}$ be a non decreasing sequence of positive numbers such that $\sum_k(\lambda_k)^{-1}$ diverges; and for $1\le p\le\infty, 1\le p(n)\uparrow p$ as $n\to\infty$. Then $f\in\Lambda BV(p(n)\uparrow p,\varphi,I)$ if

$$V_{\Lambda_{p(n)}}(f,\varphi,I)=\sup_{n\ge1}\sup_{\{I_m\}}\left\{V_{\Lambda_{p(n)}}(f,\{I_m\}):\delta\{I_m\}\ge\frac{b-a}{\varphi(n)}\right\}<\infty,$$

where $\{I_m\}$ is finite collection of non overlapping subintervals of I,

$$V_{\Lambda_{p(n)}}(f,\{I_m\})=\left(\sum_m\frac{|f(I_m)|^{p(n)}}{\lambda_m}\right)^{1/p(n)},$$

$f(I_m)=f(b_m)-f(a_m)$ and $\delta\{I_m\}=\inf_m\{|a_m-b_m|:m\in\mathbb{N}\}$.

Note that, $f\in\Lambda BV(p(n)\uparrow p,\varphi,I)$ implies f is bounded function on I [32, Lemma 3.1]. If we take $p(n)=p,\ \forall n\in\mathbb{N}$ then $\Lambda BV(p(n)\uparrow p,\varphi,I)$ coincides with $\Lambda-BV^{(p)}(I)$ [24, Definition on p. 8] for $1\le p<\infty$. If $\varphi(n)=2^n,\ n=1,2,...,$ and $\Lambda=\{1\}_1^{\infty}$ then the class $\Lambda BV(p(n)\uparrow p,\varphi,I)$ coincides with $BV(p(n)\uparrow p,I)$ [18, Definition 1.1].

Theorem 4. *If* $f\in\Lambda BV(p(n)\uparrow p,\varphi,\overline{\mathbb{T}}),\ 1\le p\le\infty$ *and* $n\in\mathbb{Z}^*$, *then*

$$\hat{f}(n)=O\left(\frac{1}{\left(\sum_{j=1}^{2|n|}\frac{1}{\lambda_j}\right)^{\frac{1}{p(\tau(|n|))}}}+\frac{1}{|n|}\right),$$

where

$$\tau(n)=\min\{k:k\in\mathbb{N},\varphi(k)\ge n\},n\ge1. \tag{4}$$

Theorem 4 is the analogous result for rational Fourier coefficients to the one obtained for the order of magnitude of the classical Fourier coefficients [32, Theorem 3].

Definition 5. [4, Definition 1] Let f be a 2π periodic measurable function. Let $p(n)$ be an increasing sequence such that $1 \le p(n) \uparrow p$ for $1 \le p \le \infty$. Then $f \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ if

$$\Lambda(f, p(n) \uparrow p, \varphi, \overline{\mathbb{T}}) = \sup_{m\geq 1} \sup_{h \geq \frac{1}{\varphi(m)}} \left\{ \frac{1}{h} \int_{\overline{\mathbb{T}}} \mid f(x+h) - f(x) \mid^{p(m)} dx \right\}^{\frac{1}{p(m)}} < \infty.$$

Note that, $f \in B\Lambda(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}})$ implies that f is essentially bounded function on $\overline{\mathbb{T}}$ [4, Corollary 1].

Theorem 6. *If $f \in B\Lambda(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}})$ and $n \in \mathbb{Z}^*$, then*

$$\hat{f}(n) = O\left(\frac{1}{\mid n \mid^{\frac{1}{p(\tau(|n|))}}} \right),$$

where

$$\tau(n) = \min\left\{ k : k \in \mathbb{N}, \varphi(k) \geq n \left(\frac{1+r}{1-r} \right) \right\}, n \geq 1. \tag{5}$$

Theorem 6 is the analogous result for rational Fourier coefficients to the one obtained for the order of magnitude of the classical Fourier coefficients [4, Theorem 5].

Chapter 3. In this chapter, we present results related to the order of magnitude of the double rational Fourier coefficients of integrable functions, functions in $Lip(p; \beta_1, \beta_2)(\overline{\mathbb{T}}^2)$ and functions with generalized bounded variations, such as $\Phi\Lambda^* BV(\overline{\mathbb{T}}^2)$, $\Lambda^* BV(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2)$ and $B\Lambda(p(t) \uparrow \infty, q(t) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$. Furthermore, these results are extended for multiple rational Fourier coefficients for functions of N variables ($N > 2$).

For a function, $f \in L^1(\overline{\mathbb{T}}^2)$, which is 2π periodic in both the variables, double rational Fourier series of f is defined as

$$f(x,y) \sim \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{f}(m,n) \phi_m(e^{ix}) \phi_n(e^{iy}),$$

where $\hat{f}(m,n)$ is the $(m,n)^{th}$ rational Fourier coefficient of f given by

$$\hat{f}(m,n) = \frac{1}{4\pi^2} \int\int_{\overline{\mathbb{T}}^2} f(x,y) \overline{\phi_m(e^{ix}) \phi_n(e^{iy})} dxdy.$$

Here, if $\alpha_k = 0, \ \forall k \in \mathbb{N}$, then above series becomes the double Fourier series of f.

Theorem 7. *If $f \in L^1(\overline{\mathbb{T}}^2)$, $(m,n) \in \mathbb{Z}^2$, then $\hat{f}(m,n) \to 0$ as $|(m,n)| \to \infty$.*

The above result is extension of [28, Theorem 2.1] for two variable functions. If we take $\alpha_k = 0, \ \forall k \in \mathbb{Z}$, we get Riemann Lebesgue lemma for double Fourier series.

Definition 8. [34, Definition-1.1] Let $f \in L^p(\overline{\mathbb{T}}^2), p \geq 1$ and $\beta_1, \beta_2 \in (0,1]$, we say that $f \in Lip(p; \beta_1, \beta_2)(\overline{\mathbb{T}}^2)$, if $\omega^{(p)}(f; \delta, \gamma) = O(\delta^{\beta_1} \gamma^{\beta_2})$ as δ and $\gamma \to 0$, where

$$\omega^{(p)}(f; \delta, \gamma) = \sup \left\{ \left(\frac{1}{4\pi^2} \int\int_{\overline{\mathbb{T}}^2} |\Delta f(x,y;h,k)|^p dxdy \right)^{\frac{1}{p}}; 0 < h \leq \delta, 0 < k \leq \gamma \right\}$$

and $\Delta f(x,y;h,k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y)$.

Theorem 9. *If* $f \in Lip(p;\beta_1,\beta_2)(\overline{\mathbb{T}}^2)$ *and* $(m,n) \in \mathbb{Z}^{*2}$ *then*

$$\hat{f}(m,n) = O\left(\frac{1}{|m|^{\beta_1}|n|^{\beta_2}} + \frac{1}{|m|} + \frac{1}{|n|}\right).$$

If we take $\alpha_k = 0,\ \forall k \in \mathbb{Z}$ in Theorem 9, then we can choose $r = 0$ and thus, $c_2 = 0$ gives us the analogous result for double Fourier coefficients [34, Theorem 2.3].

Definition 10. [35] A measurable function f defined on a rectangle $R^2 := [a,b] \times [c,d]$ is said to be of $\Phi - \Lambda-$ bounded variation (that is, $f \in \Phi\Lambda BV(R^2)$) if

$$V_{\Lambda_\Phi}(f, R^2) = \sup_{J_1,J_2}\left\{\sum_i \sum_j \frac{\Phi(|f(I_i \times K_j|)}{\lambda_{(1,i)}\lambda_{(2,j)}}\right\} < \infty,$$

where Φ is a continuous function defined on $[0,\infty)$ which is strictly increasing from 0 to ∞ such that $\Phi(0) = 0$, $\frac{\Phi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\Phi(x)}{x} \to \infty$ as $x \to \infty$; $\Lambda = (\Lambda_1, \Lambda_2)$ where $\Lambda_k = \{\lambda_{(k,n)}\}_{n=1}^{\infty}$ and $\{\lambda_{(k,n)}\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive numbers such that $\sum_n \frac{1}{\lambda_{(k,n)}}$ diverges for $k = 1,2$; J_1 and J_2 are finite collections of non-overlapping subintervals $\{I_i\}$ and $\{K_j\}$ in $[a,b]$ and $[c,d]$ respectively; and $f(I \times J) = \Delta f_{(a,c)}^{(b,d)} = f(b,d) - f(a,d) - f(b,c) + f(a,c)$.

The above generalized variation is in the sense of Vitali. Here, $f \in \Phi\Lambda BV(R^2)$ need not be bounded. This class is further generalized in the sense of Hardy as follows.

If $f \in \Phi\Lambda BV(R^2)$ is such that the marginal functions $f(.,c) \in \Phi\Lambda_1 BV([a,b])$ and $f(a,.) \in \Phi\Lambda_2 BV([c,d])$ (see [31, Definition 2]) then f is said to be of $\Phi - \Lambda^*-$ bounded variation (that is, $f \in \Phi\Lambda^* BV(R^2)$).

Note that, $f \in \Phi\Lambda^* BV(R^2)$ implies that f is bounded function in R^2 [35, cf. Proof of Corollary 1].

Here, function Φ is said to have Δ_2 condition if there exists a constant $d \geq 2$ such that $\Phi(2x) \leq d\Phi(x)$, for all $x \geq 0$.

Theorem 11. *If* $f \in \Phi\Lambda BV(\overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2)$ *and* $(m,n) \in \mathbb{Z}^{*2}$ *then*

$$\hat{f}(m,n) = O\left(\Phi^{-1}\left(\frac{|n| + |m|}{\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|}\frac{1}{\lambda_{(1,j)}\lambda_{(2,k)}}}\right)\right).$$

The above result is extension of [28, Corollary 2.6] for two variable functions. If we take $\alpha_k = 0,\ \forall k \in \mathbb{Z}$ in the above result, then we can choose $r = 0$ and thus, $c_2 = 0$ gives us analogous result for double Fourier coefficients [35, Theorem 1].

Corollary 12. *If* $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^2)$ *and* $(m,n) \in \mathbb{Z}^{*2}$ *then*

$$\hat{f}(m,n) = O\left(\Phi^{-1}\left(\frac{|n| + |m|}{\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|}\frac{1}{\lambda_{(1,j)}\lambda_{(2,k)}}}\right)\right).$$

Corollary 13. *If* Φ *satisfies* Δ_2 *condition and* $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^2)$*, then for* $m \in \mathbb{Z}^*$

$$\hat{f}(m,0) = O\left(\Phi^{-1}\left(\frac{1}{\sum_{j=1}^{2|m|}\frac{1}{\lambda_{(1,j)}}}\right)\right)$$

.

If we take $\alpha_k = 0,\ \forall k \in \mathbb{Z}$ in Corollary 12 and Corollary 13, then we can choose $r = 0$ and thus, $c_2 = 0$ gives us analogous result for double Fourier coefficients [35, Corollary 1] and [35, Corollary 2].

Definition 14. [33, Definition 2.1] Let f be a complex valued measurable function defined on $R^2 := I^{(1)} \times I^{(2)} := [a_1, b_1] \times [a_2, b_2];\ \Lambda = \left(\Lambda^{(1)}, \Lambda^{(2)}\right)$, where $\Lambda^{(t)} = \left\{\lambda_k^{(t)}\right\}_{k=1}^{\infty}$ is a non decreasing sequence of positive numbers such that $\sum_k \left(\lambda_k^{(t)}\right)^{-1}$ diverges for $t = 1, 2$; and for $1 \le p \le \infty, 1 \le p(n) \uparrow p$ as $n \to \infty$. Then $f \in \Lambda BV(p(n) \uparrow p, \varphi, R^2)$ if

$$V_{\Lambda_{p(n)}}(f, \varphi(n), R^2) = \sup_{n \ge 1} \sup_{\left\{I_i^{(1)} \times I_j^{(2)}\right\}} \left\{ V_{\Lambda_{p(n)}}\left(f, \left\{I_i^{(1)} \times I_j^{(2)}\right\}\right) : \delta\left\{I_i^{(1)} \times I_j^{(2)}\right\} \ge \frac{(b_1 - a_1)(b_2 - a_2)}{\varphi(n)^2} \right\} < \infty,$$

where $\left\{I_i^{(1)}\right\}$ and $\left\{I_j^{(2)}\right\}$ are finite collections of non overlapping subintervals of $I^{(1)}$ and $I^{(2)}$ respectively,

$$V_{\Lambda_{p(n)}}\left(f, \left\{I_i^{(1)} \times I_j^{(2)}\right\}\right) = \left(\sum_i \sum_j \frac{\left|f\left(I_i^{(1)} \times I_j^{(2)}\right)\right|^{p(n)}}{\lambda_i^{(1)}\lambda_j^{(2)}}\right)^{1/p(n)},$$

$$f\left(I^{(1)} \times I^{(2)}\right) = f(b_1, b_2) - f(a_1, b_2) - f(b_1, a_2) + f(a_1, a_2)$$

and

$$\delta\left\{I_i^{(1)} \times I_j^{(2)}\right\} := \delta\{[s_{i-1}, s_i] \times [t_{j-1}, t_j]\} = \inf_{i,j} |(s_i - s_{i-1})(t_j - t_{j-1})|.$$

Definition 15. [33, Definition 2.3] If $f \in \Lambda BV(p(n) \uparrow p, \varphi, R^2)$ and the marginal functions $f(., a_2) \in \Lambda^{(1)} BV(p(n) \uparrow p, \varphi, [a_1, b_1])$ and $f(a_1, .) \in \Lambda^{(2)} BV(p(n) \uparrow p, \varphi, [a_2, b_2])$, then $f \in \Lambda^* BV(p(n) \uparrow p, \varphi, R^2)$.

Note that, the function $f \in \Lambda^* BV(p(n) \uparrow p, \varphi, R^2)$ is bounded.

Theorem 16. *If* $f \in \Lambda^* BV(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2),\ 1 \le p \le \infty$ *and* $(m, n) \in \mathbb{Z}^{*2}$*, then*

$$\hat{f}(m,n) = O\left(\frac{1}{\left(\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|}\frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}}\right)^{\frac{1}{p(\tau(|mn|))}}} + \frac{1}{|m|} + \frac{1}{|n|}\right),$$

where $\tau(|mn|)$ *is as defined in* (4).

Theorem 17. *If $f \in \Lambda^* BV(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2),\ 1 \le p \le \infty$ and $m \in \mathbb{Z}^*$, then*

$$\hat{f}(m,0) = O\left(\frac{1}{\left(\sum_{j=1}^{2|m|} \frac{1}{\lambda_j^{(1)}}\right)^{\frac{1}{p(\tau(|m|))}}} + \frac{1}{|m|} \right),$$

where $\tau(|m|)$ is as defined in (4).

The analogous results related to the order of double Fourier coefficients [33, Theorem 3.1] and [33, Theorem 3.2] are obtained from Theorem 16 and Theorem 17, respectively, with the consideration that $r = 0$ in the rational orthogonal system.

Definition 18. Let $f \in L^\infty(\overline{\mathbb{T}}^2)$ be 2π periodic in both variables. Let $p(n)$ and $q(n)$ be increasing sequences such that $p(n) \le q(n),\ 1 \le p(n) \uparrow p$ for $1 \le p \le \infty$ and $1 \le q(n) \uparrow q$ for $1 \le q \le \infty$. Then $f \in B\Lambda(p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2)$ if

$$\Lambda(f, p(n) \uparrow p, q(n) \uparrow q, \varphi, \overline{\mathbb{T}}^2)$$

$$= \sup_{m \ge 1} \sup_{hk \ge \frac{1}{\varphi(m)^2}} \left\{ \frac{1}{k} \int_{\overline{\mathbb{T}}} \left(\frac{1}{h} \int_{\overline{\mathbb{T}}} \mid \Delta f(x,y;h,k) \mid^{p(m)} dx \right)^{\frac{q(m)}{p(m)}} dy \right\}^{\frac{1}{q(m)}} < \infty,$$

where

$$\Delta f(x,y;h,k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y).$$

Theorem 19. *If $f \in B\Lambda(p(t) \uparrow \infty, q(t) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$ and $(m,n) \in \mathbb{Z}^{*2}$, then*

$$\hat{f}(m,n) = O\left(\frac{1}{\mid m \mid^{\frac{1}{p(\tau(|mn|))}} \mid n \mid^{\frac{1}{q(\tau(|mn|))}}} + \frac{1}{\mid m \mid} + \frac{1}{\mid n \mid} \right),$$

where $\tau(mn)$ is as defined in (5).

Theorem 20. *If marginal function $f(.,b) \in B\Lambda(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}),\ b \in \overline{\mathbb{T}}$ and $m \in \mathbb{Z}^*$, then*

$$\hat{f}(m,0) = O\left(\frac{1}{\mid m \mid^{\frac{1}{p(\tau(|m|))}}} \right),$$

where $\tau(\mid m \mid)$ is defined as in (5).

If $\alpha_k = 0,\ \forall k \in \mathbb{N}$ then Theorem 19 and Theorem 20 gives analogous results for double Fourier coefficient with order containing only first term as $r = 0$, which implies $c_2 = 0$. These results for double Fourier coefficient are an extension of the result obtained by Akhobadze [4, Theorem 5] for order of Fourier coefficient.

Note that, all these results for double rational Fourier coefficients are further extended for multiple rational Fourier coefficients.

Chapter 4. In this chapter, extending the results of Fourier and conjugate Fourier series, the rates of convergence for rational and conjugate rational Fourier series for functions of generalized bounded variation are estimated.

The conjugate rational Fourier series is given by $\sum_{n=-\infty}^{\infty}(-i)\text{sgn}(n)\hat{f}(n)\phi_n(e^{ix})$. Note that, if $\alpha_k = 0,\ \forall k \in \mathbb{N}$ in (1) then $\{\phi_n(e^{ix})\}$ reduces to $\{e^{inx}\}$ and therefore conjugate rational Fourier series reduces to conjugate Fourier series.

For $n \in \mathbb{N}$ and $x \in [-\pi, \pi]$, the partial sums of rational Fourier series and conjugate rational Fourier series of f are given by

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k)\phi_k(e^{ix}) \text{ and } \tilde{S}_n f(x) = \sum_{k=-n}^{n} (-i)\text{sgn}(k)\hat{f}(k)\phi_k(e^{ix})$$

respectively.

Waterman [37] defined the concept of $\Lambda-$ bounded variation as follow.

Let $\{\lambda_n\}_{n=0}^{\infty}$ be a non decreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_n}$ diverges then a real valued function f is said to be of $\Lambda-$ bounded variation on $[a, b]$ (i.e. $f \in \Lambda BV[a, b]$) if

$$\sum_{k=1}^{n} \frac{\mid f(b_k) - f(a_k) \mid}{\lambda_k} < \infty,$$

for every sequence of non-overlapping intervals $[a_k, b_k], k = 1, ..., n,$ which is contained in $[a, b]$ and the $\Lambda-$ variation is defined by

$$V_\Lambda(f, [a, b]) = \sup \sum_{k=1}^{n} \left\{ \frac{\mid f(b_k) - f(a_k) \mid}{\lambda_k} \right\}.$$

Note that, if in above definition $\lambda_n = n^\beta;\ \beta \in (0, 1),$ then f is said to be of n^β bounded variation (i.e. $f \in \{n\}^\beta - BV[a, b]$).

As mentioned in [38], we suppose that $\frac{\lambda_{|k|}}{|k|}$ is non increasing and for fixed n, $H(t)$ is a continuously non increasing function on $[-\pi, 0)$ and $(0, \pi]$ such that $H(t) = \frac{\lambda_{|k|}}{t}$; where $t = \frac{k\pi}{n+1}$ and $k = \pm 1, \pm 2, ..., \pm(n + 1)$.

The following notations will be used in the rest of the results stated in this chapter:

(i) $\psi_x(t) = f(x) - f(x - t),\ t \in \mathbb{R},$

(ii) $\tilde{f}(x) = \lim_{\epsilon \to 0^+} \tilde{f}(x; \epsilon) = \frac{1}{\pi} \int_{\epsilon \leq \pi} \frac{f(x-t)}{2\tan(t/2)} dt,$

(iii) $osc(\psi_x, [a, b]) = \sup_{t,y \in [a,b]} |\psi_x(t) - \psi_x(y)|,$

(iv) $\eta_{km} = \frac{k\pi}{m+1},\ \forall k = 0, 1, 2, ..., m;\ m \in \mathbb{N} \cup \{0\},$

(v) $I_{km}^+ = [\eta_{km}, \eta_{(k+1)m}],$

(vi) $I_{km}^{-} = [-\eta_{(k+1)m}, -\eta_{km}]$.

First, we state the results related to the rate of convergence of rational Fourier series.

Theorem 21. *If f is bounded, measurable function in $[-\pi, \pi]$ and is regulated i.e. $f(x) = 1/2\{f(x+0) + f(x-0)\}$ then*

$$|S_n f(x) - f(x)| \le \frac{2(1+r)}{1-r} \sum_{k=0}^{n} \frac{1}{k+1} \left\{ osc(\psi_x, I_{kn}^{+}) + osc(\psi_x, I_{kn}^{-}) \right\}.$$

Corollary 22. *If $f \in \Lambda BV([-\pi, \pi])$, $\frac{\pi}{n+1} = a_n < a_{n-1} < \ldots < a_0 = \pi$ and $-\pi = b_0 < b_1, \ldots, < b_n = \frac{-\pi}{n+1}$, then*

$$\begin{aligned}\left(\frac{1-r}{1+r}\right) |S_n f(x) - f(x)| \le{}& \frac{2\lambda_{n+1}}{n+1} (V_\Lambda(\psi, [0, \pi]) + V_\Lambda(\psi, [-\pi, 0])) \\ &+ \frac{2\pi}{n+1} \sum_{i=0}^{n-1} V_\Lambda(\psi, [0, a_i])(H(a_{i+1}) - H(a_i)) \\ &+ \frac{2\pi}{n+1} \sum_{i=0}^{n-1} V_\Lambda(\psi, [b_i, 0])(H(b_i) - H(b_{i+1})).\end{aligned}$$

Corollary 23. *If $f \in \{n^\beta\} - BV([-\pi, \pi])$, $0 < \beta < 1$ then*

$$|S_n f(x) - f(x)| \le \frac{2(2-\beta)}{(n+1)^{1-\beta}} . \frac{1+r}{1-r} \sum_{k=1}^{n} \frac{1}{k^\beta} \left\{ V_{\{n^\beta\}}(\psi, [0, \pi/k]) + V_{\{n^\beta\}}(\psi, [-\pi/k, 0]) \right\}.$$

Theorem 21, Corollary 22 and Corollary 23 are analogous results of [38, 5] for rational Fourier series as for $r = 0$, the estimations for classical Fourier series are obtained. These results generalize the estimation of rational Fourier series for functions of bounded variation given by Tan and Zhou [27, Lemma 2.4]. These results are further extended for double rational Fourier series.

The similar results can be obtained for conjugate rational Fourier series which are stated below.

Theorem 24. *If f is bounded, measurable and regulated function in $[-\pi, \pi]$ then*

$$\left| \tilde{S}_n f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \le \frac{2(1+r)}{1-r} \sum_{k=0}^{n} \frac{1}{k+1} \left\{ osc(\psi_x, I_{kn}^{+}) + osc(\psi_x, I_{kn}^{-}) \right\}.$$

Corollary 25. *If $f \in \Lambda BV([-\pi, \pi])$, $\frac{\pi}{n+1} = a_n < a_{n-1} < \ldots < a_0 = \pi$ and $-\pi = b_0 < b_1, \ldots, < b_n = \frac{-\pi}{n+1}$, then*

$$\begin{aligned}\left(\frac{1-r}{1+r}\right) \left| \tilde{S}_n f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \le{}& \frac{2\lambda_{n+1}}{n+1} (V_\Lambda(\psi, [0, \pi]) + V_\Lambda(\psi, [-\pi, 0])) \\ &+ \frac{2\pi}{n+1} \sum_{i=0}^{n-1} V_\Lambda(\psi, [0, a_i])(H(a_{i+1}) - H(a_i)) \\ &+ \frac{2\pi}{n+1} \sum_{i=0}^{n-1} V_\Lambda(\psi, [b_i, 0])(H(b_i) - H(b_{i+1})).\end{aligned}$$

Corollary 26. *If* $f \in \{n^\beta\} - BV([-\pi,\pi]),\ 0<\beta<1$ *then*

$$\left|\tilde{S}_n f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right)\right| \le \frac{2(2-\beta)}{(n+1)^{1-\beta}} \cdot \frac{1+r}{1-r} \sum_{k=1}^{n} \frac{1}{k^\beta} \left\{ V_{\{n^\beta\}}(\psi, [0, \pi/k]) + V_{\{n^\beta\}}(\psi, [-\pi/k, 0]) \right\}.$$

Corollary 26 generalizes the result obtained in [27, Lemma 2.4]. If $r=0$, then Theorem 24 gives estimation for conjugate Fourier series.

Chapter 5. In this chapter, it is obtained that the rational trigonometric series with its coefficients $c(n)=o(1)$ satisfying the condition $\sum_{n\in\mathbb{Z}} |\Delta^m c(n)| < \infty,\ m\in\mathbb{N}$, converges pointwise to some $f(x)$ for every $x \in (0,2\pi)$ and also converges in $L^p[0,2\pi)$-metric to f for $0<p<\frac{1}{m}$. This result is further extended for double rational trigonometric series. In this chapter, we consider the rational orthogonal system $\{\phi_n(e^{ix})\}_{n=-\infty}^{\infty}$, which is given below:

$$\phi_0(e^{ix}) = 1,\ \phi_n(e^{ix}) = \frac{\sqrt{1-r^2}e^{ix}}{1-re^{ix}} \left(\frac{e^{ix}-r}{1-re^{ix}}\right)^{n-1},\ \phi_{-n}(e^{ix}) = \overline{\phi_n(e^{ix})}, \tag{6}$$

where $n\in\mathbb{N},\ x\in\mathbb{T}$ and $r\in[0,1)$.

Clearly, (6) is obtained from (1) by putting $\alpha_k = r,\ \forall n \in \mathbb{N}$. If $r=0$, then the above rational orthogonal system $\{\phi_n(e^{ix})\}_{n=-\infty}^{\infty}$ reduces to the exponential system $\{e^{inx}\}_{n=-\infty}^{\infty}$.

The rational trigonometric series is defined as

$$\sum_{n=-\infty}^{\infty} c(n)\ \phi_n(e^{ix}), \tag{7}$$

where $\{c(n)\}_{n=1}^{\infty}$ is a sequence of complex numbers and $x\in\mathbb{T} := [0,2\pi)$.

Note that, if $r=0$, then the above rational trigonometric series reduces to the classical trigonometric series.

Definition 27. [25] A null sequence of complex numbers $\{c(n)\}_{n=-\infty}^{\infty}$ is said to be of bounded variation of order m (denoted by $\mathcal{BV}^m$), $m\in\mathbb{N}$ if

$$\sum_{n=-\infty}^{\infty} |\Delta^m c(n)| < \infty,$$

where $\Delta^m c(n) = \Delta^{m-1}c(n) - \Delta^{m-1}c(n+1)$ and $\Delta^0 c(n) = c(n)$.

Theorem 28. *If for some* $m\in\mathbb{N}$, *a complex sequence* $\{c(n)\}_{n\in\mathbb{Z}} \in \mathcal{BV}^m$ *then the rational trigonometric series* (7)

(i) *converges pointwise to some function* $f(x)$ *for every* $x\in\mathbb{T}\setminus\{0\}$.

(ii) *converges in* $L^p(\mathbb{T})$*-metric to* f *for any* $0<p<\frac{1}{m}$.

If $r = 0$ in Theorem 28, then we get the analogous result for classical trigonometric series obtained by Stanojevic [25], as the rational orthogonal system $\{\phi_n(e^{ix})\}_{n=-\infty}^{\infty}$ reduces to the exponential system $\{e^{inx}\}_{n=-\infty}^{\infty}$. Similarly, for $r = 0$ and $m = 1$, we get the analogous result by Ul'yanov[30].

The double rational trigonometric series is defined as

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c(j,k)\phi_j(e^{ix})\phi_k(e^{iy}), \tag{8}$$

where $\{c(j,k) : -\infty < j,k < \infty\}$ is a double sequence of complex numbers and $(x,y) \in \mathbb{T}^2$.

The rectangular partial sums of double rational trigonometric series (8) is given by

$$S_{JK}(x,y) = \sum_{|j|\le J} \sum_{|k|\le K} c(j,k)\phi_j(e^{ix})\phi_k(e^{iy}). \tag{9}$$

The series (8) is said to converge in Pringsheim's sense to $f(x,y)$ if $S_{JK}(x,y) \to f(x,y)$ as $\min(J,K) \to \infty$. In addition, if the row series $\sum_{j=-\infty}^{\infty} c(j,k)\phi_j(e^{ix})\phi_k(e^{iy})$ converges for each fixed value of k and the column series $\sum_{k=-\infty}^{\infty} c(j,k)\phi_j(e^{ix})\phi_k(e^{iy})$ converges for each fixed value of j then the double rational trigonometric series (8) is said to converge regularly to $f(x,y)$ [14].

Chen and Wu [8] defined the following bounded variation of higher order for double sequences.

Definition 29. A double sequence of complex numbers $\{c(j,k)\}_{(j,k)\in\mathbb{Z}^2}$ is said to be of bounded variation of order m (denoted by $\mathcal{BV}_2^m$) if $c(j,k) \to 0$ as $\max(|j|,|k|) \to \infty$ and for $m \in \mathbb{N}$,

$$\lim_{|k|\to\infty} \sum_{j=-\infty}^{\infty} |\Delta_{m0}c(j,k)| = 0,\ \lim_{|j|\to\infty} \sum_{k=-\infty}^{\infty} |\Delta_{0m}c(j,k)| = 0$$

$$\text{and} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{mm}c(j,k)| < \infty;$$

where

$$\Delta_{00}c(j,k) = c(j,k),\ \Delta_{mn}c(j,k) = \Delta_{m-1,n}c(j,k) - \Delta_{m-1,n}c(j+1,k)\ (m \ge 1)$$

and

$$\Delta_{mn}c(j,k) = \Delta_{m,n-1}c(j,k) - \Delta_{m,n-1}c(j,k+1)\ (n \ge 1).$$

Here,

$$\Delta_{mn}c(j,k) = \sum_{p=0}^{m} \sum_{q=0}^{n} (-1)^{p+q} \binom{m}{p} \binom{n}{q} c(j+p,k+q). \tag{10}$$

Theorem 30. *If for some $m \in \mathbb{N}$, a double complex sequence $\{c(j,k)\}_{(j,k)\in\mathbb{Z}^2} \in \mathcal{BV}_2^m$, then the double rational trigonometric series* (8)

(i) *converges regularly to some function* $f(x, y)$ *for every* $(x, y) \in (\mathbb{T} \setminus \{0\})^2$.

(ii) *converges in* $L^p(\mathbb{T}^2)$*-metric to* f *for any* $0 < p < \frac{1}{m}$ *when* $\min(j, k) \to \infty$.

Note that, if we take $r = 0$ in Theorem 30 then we get results for classical double trigonometric series. Results for double trigonometric series for bounded variation of some order m and order 1 can be found in [17] and [21] respectively.

A brief version of the material covered in chapters 1 through 5 is present in the following research articles:

1. H. J. Khachar and R. G. Vyas. A note on multiple rational Fourier series, **Periodica Mathematica Hungarica**, (2022), 85(2), 264–274, ISSN: 0031-5303. **(Scopus, SCIE)**. DOI: 10.1007/s10998-021-00433-7
2. H. J. Khachar and R. G. Vyas. Rate of convergence for rational and conjugate rational Fourier series of functions of generalized bounded variation, **Acta et Commentationes Universitatis Tartuensis de Mathematica (ACUTM)**, (2022), 26(2), 233–241, ISSN: 1406-2283. **(Scopus, ESCI)**. DOI: 10.12697/ACUTM.2022.26.16
3. H. J. Khachar and R. G. Vyas. Properties of rational Fourier series and generalized Wiener class, **Georgian Mathematical Journal**, (2023), 30(2), 247–253, ISSN: 1072-947X. **(Scopus, SCIE)**. DOI: 10.1515/gmj-2022-2206
4. H. J. Khachar and R. G. Vyas. Order of multiple rational Fourier coefficients for functions of generalized Wiener class, **Publicationes Mathematicae Debrecen**, ISSN: 0033-3883 (Accepted). **(Scopus, SCIE)**
5. H. J. Khachar and R. G. Vyas. Convergence and integrability of rational and double rational trigonometric series with coefficients of bounded variation of higher order, **Georgian Mathematical Journal**, ISSN: 1072-947X (Accepted). **(Scopus, SCIE)**
6. H. J. Khachar and R. G. Vyas. Rate of convergence for double rational Fourier series of functions of generalized bounded variation (Communicated).

The following list contains the research work presented in conferences:

1. H. J. Khachar. Order of multiple rational Fourier coefficients of functions of Akhobadze class, 87th Annual Conference of the **Indian Mathematical Society** was organized by MGM University, Aurangabad during December 4-7, 2021.
(For the above work, I was awarded **V. M. Shah Prize for the year 2021** by the Indian Mathematical Society for presenting the best research paper)
2. H. J. Khachar. Order of rational and multiple rational Fourier coefficients of functions of Φ - bounded variation, The International Conference on Mathematical Analysis and Applications was organized by **National Institute of Technology, Tiruchirappalli (NIT, Trichy)** during December 15-17, 2022.
(Joint work with Prof. R. G. Vyas)

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