STUDY OF RATIONAL FOURIER SERIES

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Chapter 1

Introduction

The classical Fourier series is generalized to various orthogonal Fourier series. These series, like the Fourier series, Legendre Fourier series, Chebychev Fourier series and Walsh Fourier series, are each tailored to deal with specific types of functions, such as smooth periodic functions [11], smooth bounded functions [6], analytic functions [28] and binary functions [10] respectively. Depending on the characteristics of the function being approximated, different orthogonal series offer different benefits in terms of convergence properties. However, when dealing with non-periodic functions or those with discontinuities or singularities, the classical Fourier series often is not very useful. In such cases, the rational Fourier series is a better alternative [9, 5]. With the right parameter adjustments, the rational Fourier series can converge faster and provide more accurate results compared to the classical Fourier series for certain functions [20, 21]. It's worth noting though, that computing the rational Fourier series may require more complexity and computational resources than its classical counterpart. Besides theoretical applications [13, 19, 22], the rational Fourier series finds numerous other applications in fields such as control theory [4], system identification [1], signal compression [14], denoising [31], and many more fields. Understanding how the rational Fourier series behaves, especially considering its use of the rational orthogonal system, is crucial due to its widespread applications and potential advantages over the classical Fourier series.

The rational Fourier series [7] is defined using the rational orthogonal system (also referred as Malmquist-Takenaka system [15, 25]), which is characterized by a complex sequence of poles within the open unit disk. The completeness condition for this system is satisfied by ensuring the poles remain within a certain bound. The rational Fourier series reduces to the classical Fourier series when all poles are zero. Specific properties of the rational Fourier series, such as its behaviour under the convolution and the magnitude of its coefficients, differ from those of the classical Fourier series. For instance, convolution of functions does not follow the same relation for rational Fourier coefficients as it does for classical Fourier coefficients. Additionally, bounds on the magnitude of coefficients differ between the two series. Thus, it is interesting to know which properties differs between rational Fourier series and classical Fourier series.

Chapter 2

Order of magnitude of rational Fourier coefficients

The study of the Fourier series traces its origin back to the early 19th century and has continually expanded due to its theoretical and practical implications. A significant contribution to this field is the Riemann Lebesgue Lemma, which laid groundwork for understanding the relationship between Fourier coefficients and the behavior of the studied function [8, Lemma 2.3.8, p. 36]. However, this lemma does not specify a definitive rate at which Fourier coefficients tend to zero; in fact, they can approach zero as slowly as desired. Consequently, mathematicians began investigating this property for various subclasses of $L^1(\overline{\mathbb{T}})$, where $\mathbb{T} = [0, 2\pi)$.

Schramm and Waterman [23, p. 408] obtained the result for the order of Fourier coefficient for functions of $\Phi - \Lambda -$ bounded variation. Tan and Zhou [27] carried out the study of rational Fourier coefficients in 2013. Firstly, they gave an analogous result of Riemann Lebesgue Lemma for rational Fourier series. Also, the result of Schramm and Waterman of the order of Fourier coefficients of functions of $\Phi\Lambda BV[0, 2\pi]$ is generalized for rational Fourier coefficient.

In 2002, Akhobadze [2] proposed another notion of generalized bounded variation, termed $B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$, and provided results regarding the order of Fourier coefficients for this class. Additionally, in 2011, a concept of generalized bounded variation, $\Lambda BV(p(n) \uparrow p, \varphi, I)$, was introduced [30], and an estimation of the order of magnitude of Fourier coefficients was made for this variation. One another subclass of $L^1([0, 2\pi])$, that was explored for the study of the magnitude order of Fourier coefficients is $Lip(\beta, p)([0, 2\pi])$ class [33, see p. 45]. The result concerning the order of Fourier coefficients for functions in $Lip(\beta, p)([0, 2\pi])$ class was obtained. In this chapter, we generalize all these results for rational Fourier coefficients and thus obtained the order of Fourier coefficients for functions in $B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$, $\Lambda BV(p(n) \uparrow p, \varphi, I)$ and $Lip(\beta, p)([0, 2\pi])$ classes.

Chapter 3

Order of magnitude of double and multiple rational Fourier coefficients

In the previous chapter, we worked on the magnitudes of rational Fourier coefficients across a range of generalized bounded variation classes. This idea was obtained from earlier research and by examining the magnitudes of classical Fourier coefficients.

In 2002, Móricz [18] extended the results of Fourier coefficients to functions of two variables, particularly those with bounded variation in the Hardy sense. This paved the way for exploring the magnitudes of double Fourier coefficients across various functions in two dimensions. While many researchers contributed to this, we noticed a gap: not much study had been done on the magnitudes of double rational Fourier coefficients. This observation inspired our research in this area.

In our study, we defined the double rational Fourier series and investigated the magnitudes of double rational Fourier coefficients for functions with generalized bounded variation in two variables, considering both Vitali and Hardy senses. Therefore, we extended upon the findings from our previous chapter on one-variable rational Fourier coefficients and applied them to double rational Fourier coefficients. We also stated the results for the order of magnitude of multiple rational Fourier coefficients.

Chapter 4

Rate of convergence of rational, conjugate rational and double rational Fourier series

The convergence rate of Fourier series relies on the function's smoothness, crucial for accurately representing functions as infinite sums of sines and cosines. Analysing convergence of Fourier series is pivotal across mathematics, engineering, physics, and signal processing. Notably, the Dirichlet-Jordan test assesses convergence for functions of bounded variation, extending its applicability beyond theoretical mathematics.

In 1971, Bojanić [3, p. 57] developed a quantitative version of the Dirichlet-Jordan test based on variations. Subsequently, in 1982, Waterman [32, p. 52] gave an estimate for the convergence rate of Fourier series, particularly for functions closed to harmonic bounded variation. In 1987, Mazhar and Al-Budaiwi [16, p. 178] derived an estimate for the convergence rate of conjugate Fourier series for functions of bounded variation. Móricz [17, Theorem 3, p. 349] extended the quantitative version of the Dirichlet-Jordan test to double Fourier series in 1992. Finally, in 2013, Tan and Qian [26, Theorem 2.4, p. 545] obtained a similar quantitative version of the Dirichlet-Jordan test for rational and conjugate rational Fourier series. These contributions served as inspiration for our investigation and thus we obtained the convergence rates of rational, conjugate rational, and double rational Fourier series for functions with generalized bounded variations in this chapter.

Chapter 5

Convergence and integrability of rational and double rational trigonometric series

Complex numbers and bounded variation sequences are vital for analysing convergence of trigonometric series and behaviour of function, essential in mathematical analysis. They ensure accurate approximation of function.

In 1954, Ul'yanov [29] obtained that sine and cosine series converge in $L^p[0, 2\pi)$ for any $0 < p < 1$, if their coefficients form a null sequence of bounded variation. Stanojevic [24] further explored convergence and integrability of complex trigonometric series with coefficients of generalized bounded variation in 1984. Later, Kaur et al. [12] extended these results to double trigonometric series in 2004.

Motivated by the importance of bounded variation sequences in analysing convergence and integrability of trigonometric series, we defined the rational trigonometric series which reduces to trigonometric series when the poles are zero in the orthogonal system and obtained results for convergence and integrability of rational trigonometric series analogous of the result obtained by Stanojevic [24] for trigonometric series. Later on, this results were extended for double trigonometric series and analogous results to that of Kaur et. al [12] was obtained.

Conclusion

The rational Fourier series can be seen as a generalization of the classical Fourier series, with the classical series emerging when the poles in the rational orthogonal system are considered as zero. In this thesis, we have derived results concerning the order of magnitude of rational Fourier coefficients for functions from various generalized bounded variation classes. These findings have been further extended to encompass multiple rational Fourier coefficients. Notably, these results differs from their classical counterparts. Additionally, we have determined the convergence rates for rational, conjugate rational, and double rational Fourier series for functions of generalized bounded variations. Understanding these convergence rates is crucial for analysing the behaviour of functions, and it has been demonstrated that by selecting appropriate poles, the rational Fourier series can converge more rapidly than the classical Fourier series. Moreover, we have established results regarding the convergence and integrability of rational and double rational trigonometric series, which closely resemble their classical counterparts concerning convergence and integrability of trigonometric and double trigonometric series.

Published/Accepted Research Articles

1. H. J. Khachar and R. G. Vyas. A note on multiple rational Fourier series, **Periodica Mathematica Hungarica**, (2022), 85(2), 264–274, ISSN: 0031-5303. **(Indexed in Scopus and SCIE)**. MR4514182
DOI: 10.1007/s10998-021-00433-7

2. H. J. Khachar and R. G. Vyas. Rate of convergence for rational and conjugate rational Fourier series of functions of generalized bounded variation, **Acta et Commentationes Universitatis Tartuensis de Mathematica (ACUTM)**, (2022), 26(2), 233–241, ISSN: 1406-2283. **(Indexed in Scopus and ESCI)**. MR4527802
DOI: 10.12697/ACUTM.2022.26.16

3. H. J. Khachar and R. G. Vyas. Properties of rational Fourier series and generalized Wiener class, **Georgian Mathematical Journal**, (2023), 30(2), 247–253, ISSN: 1072-947X. **(Indexed in Scopus and SCIE)**. MR4565959
DOI: 10.1515/gmj-2022-2206

4. H. J. Khachar and R. G. Vyas. Order of multiple rational Fourier coefficients for functions of generalized Wiener class, **Publicationes Mathematicae Debrecen**, (2023), 103(3–4), 473-487, ISSN: 0033-3883. **(Indexed in Scopus and SCIE)**. MR4672947
DOI: 10.5486/PMD.2023.9550

5. H. J. Khachar and R. G. Vyas. Convergence and integrability of rational and double rational trigonometric series with coefficients of bounded variation of higher order, **Georgian Mathematical Journal**, (2023), 30(5),

739–744. ISSN: 1072-947X. **(Indexed in Scopus and SCIE)**. MR4649200 DOI: 10.1515/gmj-2023-2041

6. H. J. Khachar and R. G. Vyas. Rate of convergence for double rational Fourier series, **Complex Analysis and Operator Theory**, ISSN: 1661-8254 **(Accepted)**. **(Indexed in Scopus, SCIE)**

Presented Research Work in Conferences

1. H. J. Khachar. Order of multiple rational Fourier coefficients of functions of Akhobadze class, 87th Annual Conference of the **Indian Mathematical Society** was organized by MGM University, Aurangabad during December 4-7, 2021.
   (For the above work, **V. M. Shah Prize for the year 2021** was awarded by the Indian Mathematical Society for presenting the best research paper)

2. H. J. Khachar. Order of rational and multiple rational Fourier coefficients of functions of Φ - bounded variation, The International Conference on Mathematical Analysis and Applications was organized by **National Institute of Technology, Tiruchirappalli (NIT, Trichy)** during December 15-17, 2022.
   (Joint work with Prof. R. G. Vyas)

Bibliography

[1] H. Akçay. On the uniform approximation of discrete-time systems by generalized Fourier series. *IEEE transactions on signal processing*, 49(7):1461–1467, 2001.

[2] T. Akhobadze. A generalization of bounded variation. *Acta Mathematica Hungarica*, 97(3):223–256, 2002.

[3] R. Bojanić. An estimate of the rate of convergence for Fourier series of functions of bounded variation. *Publ. Inst. Math. (Beograd) (N.S.)*, 26 (40):57–60, 1979.

[4] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njastad, et al. *Orthogonal rational functions*. Cambridge Monographs on Applied and Computational Mathematics, 5. Cambridge University Press, Cambridge, 1999.

[5] Qiuhui Chen, Tao Qian, Yuan Li, Weixiong Mai, and Xingfa Zhang. Adaptive Fourier tester for statistical estimation. *Math. Methods Appl. Sci.*, 39(12):3478–3495, 2016.

[6] Harry F. Davis. *Fourier series and orthogonal functions*. New York: Dover Publications, Inc., reprint of the 1963 ed. edition, 1989.

[7] M. M. Džrbašyan. On the theory of series of Fourier in terms of rational functions. *Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki*, 9:3–28, 1956.

[8] R. E. Edwards. *Fourier series. A modern introduction. Vol. 1*, volume 64 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1979.

[9] Timea Eisner and Margit Pap. Discrete orthogonality of the Malmquist Takenaka system of the upper half plane and rational interpolation. *J. Fourier Anal. Appl.*, 20(1):1–16, 2014.

[10] B. Golubov, A. Efimov, and V. Skvortsov. *Walsh series and transforms*, volume 64 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. Theory and applications, Translated from the 1987 Russian original by W. R. Wade.

[11] David W. Kammler. *A first course in Fourier analysis*. Cambridge University Press, Cambridge, second edition, 2007.

[12] Kulwinder Kaur, S. S. Bhatia, and Babu Ram. Double trigonometric series with coefficients of bounded variation of higher order. *Tamkang J. Math.*, 35(3):267–280, 2004.

[13] Natallia Yu. Kazlouskaya and Yaugeni A. Rovba. Approximation of the function $|\mathrm{SIN}\, x|^s$ by the partial sums of the trigonmometric rational Fourier series. *Dokl. Nats. Akad. Nauk Belarusi*, 65(1):11–17, 2021.

[14] J. Ma, T. Zhang, and M. Dong. A Novel ECG Data Compression Method Using Adaptive Fourier Decomposition With Security Guarantee in e-Health Applications. *IEEE Journal of Biomedical and Health Informatics*, 19(3):986–994, 2015.

[15] Folke Malmquist. Sur la détermination d'une classe de fonctions analytiques par leurs valeurs dans un ensemble donné de points. In *Comptes Rendus du Sixieme Congres*, pages 253–259, 1925.

[16] S. Mazhar and A. Al-Budaiwi. An estimate of the rate of convergence of the conjugate Fourier series of functions of bounded variation. *Acta Math. Hungar.*, 49(3-4):377–380, 1987.

[17] Ferenc Móricz. A quantitative version of the Dirichlet-Jordan test for double Fourier series. *J. Approx. Theory*, 71(3):344–358, 1992.

[18] Ferenc Móricz. Order of magnitude of double Fourier coefficients of functions of bounded variation. *Analysis (Munich)*, 22(4):335–345, 2002.

[19] Margit Pap and Ferenc Schipp. Equilibrium conditions for the Malmquist-Takenaka systems. *Acta Sci. Math. (Szeged)*, 81(3-4):469–482, 2015.

[20] Gerlind Plonka and Vlada Pototskaia. Computation of adaptive Fourier series by sparse approximation of exponential sums. *J. Fourier Anal. Appl.*, 25(4):1580–1608, 2019.

[21] Tao Qian, Liming Zhang, and Zhixiong Li. Algorithm of adaptive Fourier decomposition. *IEEE Trans. Signal Process.*, 59(12):5899–5906, 2011.

[22] V. V. Savchuk. Best approximations of the Cauchy-Szegö kernel in the mean on the unit circle. *Ukr. Math. J.*, 70(5):817–825, 2018.

[23] Michael Schramm and Daniel Waterman. On the magnitude of Fourier coefficients. *Proc. Am. Math. Soc.*, 85:407–410, 1982.

[24] Vera B. Stanojevic. On a theorem of P. L. Uljanov. *Proc. Amer. Math. Soc.*, 90(3):370–372, 1984.

[25] S. Takenaka. On the orthogonal functions and a new formula of interpolation. In *Japanese journal of mathematics: transactions and abstracts*, volume 2, pages 129–145. The Mathematical Society of Japan, 1925.

[26] L. Tan and T. Qian. On convergence of rational Fourier series of functions of bounded variations (in Chinese). *Scientia Sinica Mathematica*, 43(6):541–550 https://doi.org/10.1360/012013–127, 2013.

[27] L. Tan and C. Zhou. On the order of rational Fourier coefficients of various bounded variations. *Complex Variables and Elliptic Equations*, 58(12):1737–1743, 2013.

[28] Lloyd N. Trefethen. *Approximation theory and approximation practice.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013.

[29] P. L. Ul'yanov. Application of A-integration to a class of trigonometric series. *Mat. Sb. N.S.*, 35(77):469–490, 1954.

[30] R. G. Vyas. A note on functions of $p(n) - \Lambda-$bounded variation. *J. Indian Math. Soc*, 78(1-4):215–220, 2011.