Chapter 1

Introduction

The Fourier series of a 2π periodic and integrable function f is defined as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n(f) e^{inx}, \tag{1.1}$$

where $c_n(f)$ is the n^{th} Fourier coefficient of f, given by

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx. \tag{1.2}$$

The study of the Fourier series traces its origin to the early 19th century and has been an ever-growing topic since then due to its theoretical and practical applications. The significant result called Riemann Lebesgue Lemma [20, Lemma 2.3.8, p. 36], laid some groundwork for determining the link between the Fourier coefficients with the behaviour of the function being studied and it can be stated as follows:

$$\text{If } f \in L^1([0, 2\pi]),\ n \in \mathbb{Z} \text{ then } c_n(f) \to 0 \text{ as } |n| \to \infty.$$

It is observed that the Riemann-Lebesgue Lemma does not provide a definite rate at which the Fourier coefficients tend to zero; in fact, the Fourier coefficients can tend to zero as slowly as desired. Therefore, some mathematicians started studying properties of Fourier coefficients for various subclasses of $L^1(\overline{\mathbb{T}}),\ \mathbb{T} = [0, 2\pi)$.

1.1 Fourier coefficients' properties of one variable functions of generalized bounded variations

One specific subclass of $L^1([0, 2\pi])$, that was explored for the study of the magnitude order of Fourier coefficients is $Lip(\beta, p)([0, 2\pi])$ class [79, p. 45].

Definition 1.1.1. Let $f \in L^p(\overline{\mathbb{T}}), p \geq 1$ and $\beta \in (0, 1]$, then $f \in Lip(p; \beta)(\overline{\mathbb{T}})$, if $\omega^{(p)}(f; \gamma) = O(\gamma^\beta)$ as $\gamma \to 0$, where

$$\omega^{(p)}(f; \gamma) = \sup_{0<h\leq\gamma} ||T_h(f) - f||_p,$$

$T_h(f)(x) = f(x + h),\ \forall h, x \in \overline{\mathbb{T}}$ and $||f||_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\right)^{\frac{1}{p}}$.

If $p = \infty$ then $Lip(p; \beta)(\overline{\mathbb{T}})$ reduces to $Lip(\beta)(\overline{\mathbb{T}})$.

The result [79, Theorem 4.7, p.46] concerning the order of Fourier coefficients for functions in $Lip(\beta, p)([0, 2\pi])$ class was shown as follows:

Theorem A. If $f \in Lip(\beta, p)([0, 2\pi])$ and $n \in \mathbb{Z} \setminus \{0\}$ then

$$c_n(f) = O\left(\frac{1}{|n|^\beta}\right).$$

One another subclass of $L^1([0, 2\pi])$ called a class of functions of bounded variations $BV([0, 2\pi])$ was first introduced by Jordan [30] in 1881. The Jordan class of bounded variation is quite useful not only in the study of the order of Fourier coefficients but also in the study of other aspects, such as pointwise and absolute convergence of Fourier series. Since the class is Banach algebra with respect to the pointwise operations and suitable variation norm, it also finds theoretical applications in functional analysis. Mathematicians' pursuit of elegance and/or generality in addressing specific problems has resulted in fascinating generalizations of the concept of bounded variation. This has led to the emergence of novel categories of functions with generalized bounded variations.

In 1924, Wiener [77] introduced the class of functions of bounded variation of order p, $p \geq 1$, written as $BV^p([0, 2\pi])$. Later on, influenced by the study of problems of the convergence of Fourier series, Waterman [75] gave the class of functions of $\Lambda-$ bounded variation, written as $\Lambda BV([0,2\pi])$. Subsequently, in 1980, Shiba [57] gave the following class of functions of $p-\Lambda-$bounded variation by assimilating ideas from Wiener's bounded variation of order p and Waterman's $\Lambda-$ bounded variation.

Definition 1.1.2. Let $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ be a non decreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_n}$ diverges and p be a real number such that $1 \leq p < \infty$ then a real-valued function f is said to be of $p-\Lambda-$ bounded variation on $[a,b]$ (i.e. $f \in \Lambda BV^p([a,b])$) if

$$V_{\Lambda_p}(f,[a,b]) = \sup_{\{I_n\}} V_{\Lambda_p}(\{I_n\}, f, [a,b]) < \infty,$$

where

$$V_{\Lambda_p}(\{I_n\}, f, [a,b]) = \left(\sum_{k=1}^{n} \frac{|f(I_k)|^p}{\lambda_k} \right)^{\frac{1}{p}} < \infty,$$

for every sequence of non-overlapping intervals $I_k := [a_k, b_k]$ which is contained in $[a,b]$ and $f(I_k) = f(b_k) - f(a_k)$ for $k = 1, ..., n,$.

Note that, in the above definition, if we substitute

- $p = 1$ and $\Lambda = \{1\}_{n=1}^{\infty}$ then $\Lambda BV^p([a,b])$ reduces to Jordan variation $BV([a,b])$.
- $\Lambda = \{1\}_{n=1}^{\infty}$ then $\Lambda BV^p([a,b])$ reduces to Wiener variation $BV^p([a,b])$.
- $p = 1$ then $\Lambda BV^p([a,b])$ reduces to Waterman variation $\Lambda BV([a,b])$ and if in addition $\Lambda = \{n\}_{n=1}^{\infty}$ then $\Lambda BV^p([a,b])$ reduces to harmonic bounded variation $HBV([a,b])$.

It can be seen [45, p. 900] that the condition imposed on the sequence $\{\lambda_n\}_{n=0}^{\infty}$ ensures that $BV([a,b]) \subset \Lambda BV([a,b])$, but $\Lambda BV([a,b])$ does not contain all bounded functions.

Clearly,

$$BV([a,b]) \subset BV^p([a,b]) \subset \Lambda BV^p([a,b]) \tag{1.3}$$

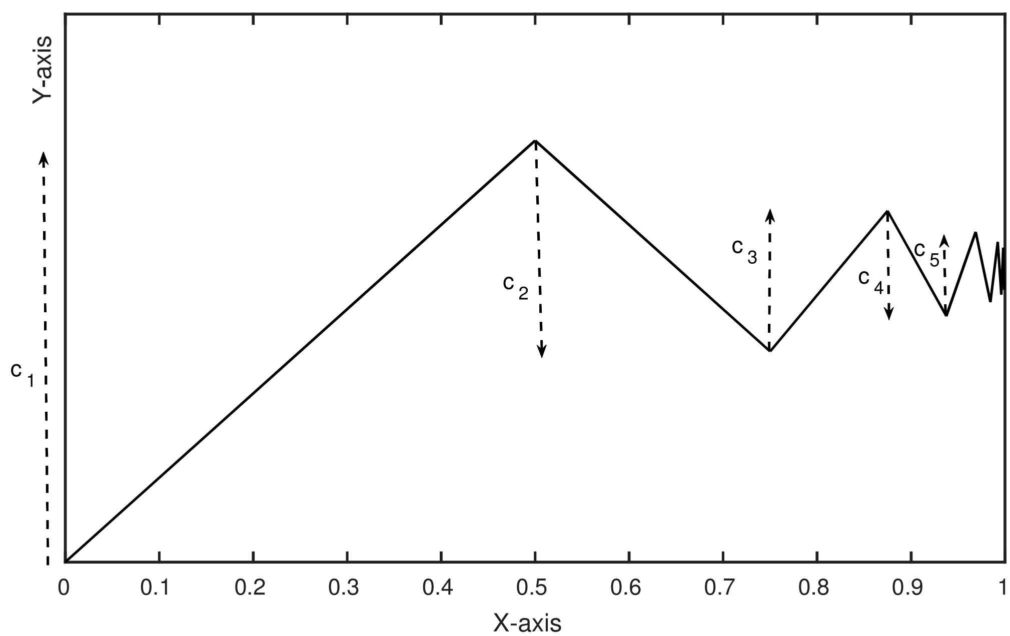
follows from

$$\left(\sum_{k=1}^{n} \frac{|f(I_k)|^p}{\lambda_k}\right)^{\frac{1}{p}} \leq \left(\frac{1}{\lambda_1}\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |f(I_k)|^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{\lambda_1}\right)^{\frac{1}{p}} \sum_{k=1}^{n} |f(I_k)|,$$

where $f(I_k)$, p and $\{\lambda_k\}_k$ are as defined in Definition 1.1.2. The inclusion in (1.3) is proper and can be shown using the zigzag function.

Example 1.1.1. Let $c = \{c_n\}$ be a non-increasing sequence of positive numbers such that it converges to 0 as $n \to \infty$. Let the function $G(x) := G(c; x)$ be defined as continuous function on $[0, 1]$ with $G(0) = 0$ such that $G(x)$ linearly increases by c_1 on $\left[0, \frac{1}{2}\right]$, linearly decreases by c_2 on $\left[\frac{1}{2}, \frac{3}{4}\right]$, linearly increases by c_3 on $\left[\frac{3}{4}, \frac{7}{8}\right]$ and so on. Clearly, $G(1) = \sum_{n=1}^{\infty}(-1)^{n+1}c_n$.

This function G is called a zigzag function due to the zigzag nature of the function's graph, as seen below. In the graph, the solid lines represent the graph of the zigzag function and dashed lines represent c_n's.



In view of [24, cf. p. 1069], we have,

$$V_{\Lambda_p}(G,[0,1]) = \sum_{n=1}^{\infty} \frac{c_n^p}{\lambda_n}.$$

Thus, for $c_n = \frac{1}{n}$, $G(x) \notin BV([0,1])$ but $G(x) \in BV^2([0,1])$ and for $c_n = \frac{1}{\sqrt{n}}$, $G(x) \notin BV^2([0,1])$ but $G(x) \in \{n\}BV^2([0,1])$.

Pierce and Velleman [45, cf. 901] showed that the following inclusion properties hold for $1 < p_1 < p_2 < \infty$ and $0 < \beta_1 < \beta_2 \leq 1$:

$$BV^{p_1}([a,b]) \subset BV^{p_2}([a,b])$$

and

$$\{n^{\beta_1}\}BV([a,b]) \subset \{n^{\beta_2}\}BV([a,b]).$$

These properties were further generalized by Goodarzi et al. [39, Corollary 1.5, p. 831] as below: $1 < p_1 < p_2 < \infty$ and $0 < \beta_1 < \beta_2 \leq 1$:

$$\Lambda BV^{p_1}([a,b]) \subset \Lambda BV^{p_2}([a,b])$$

and

$$\{n^{\beta_1}\}BV^{p_1}([a,b]) \subset \{n^{\beta_2}\}BV^{p_1}([a,b]).$$

Vyas [70, Theorem 2, p. 731] observed that $(\Lambda BV^p([a,b]), ||.||_{\Lambda_p})$ is a commutative unital Banach algebra with respect to the pointwise operations and the following norm

$$||f||_{\Lambda_p} = ||f||_\infty + V_{\Lambda_p}(f,[a,b]) \;\; f \in \Lambda BV^p([a,b]).$$

Let $R([a,b])$ be a class of regular functions, i.e., bounded functions which have, at most, removable discontinuities or discontinuities of the first kind (jumps) in $[a,b]$. It was observed in [65, cf. p. 92], that the following inclusion properties hold:

$$BV^p([a,b]) = \bigcap_{\Lambda} \Lambda BV^p([a,b])$$

and

$$\bigcup_{\Lambda} \Lambda BV^p([a,b]) = R([a,b]).$$

In 2012, Vyas and Darji [15, Definition 2.1, p. 182] gave a definition of $p-\Lambda$ variation for a function $f:\sigma\to\mathbb{B}$ denoted by $\Lambda BV^p(\sigma,\mathbb{B})$, where σ is any non-empty compact subset of $\mathbb{R}$ and $\mathbb{B}$ is a Banach algebra. In view of [15, Theorem 2.6, p. 183], for a commutative Banach algebra $\mathbb{B}$, $\Lambda BV^p(\sigma,\mathbb{B})$ with suitable variation norm is a commutative Banach algebra with respect to pointwise operations.

Schramm and Waterman [55, p. 408] obtained the following result for the order of Fourier coefficient for functions of Λ- bounded variation.

Theorem B. If $f\in\Lambda BV^p([0,2\pi])$, $p\geq 1$ and $n\in\mathbb{N}$, then

$$c_n(f)=O\left(\left(\frac{1}{\sum_{j=1}^{n}\frac{1}{\lambda_j}}\right)^{\frac{1}{p}}\right).$$

In 1937, Young [78] generalized the notion of bounded variation of order p, $p\geq 1$ and introduced the class of functions of Φ bounded variation, written as $\Phi BV([0,2\pi])$. In 1982, Schramm and Waterman [55] gave the following generalization of bounded variation by assimilating ideas of Young's Φ variation and Waterman's Λ variation.

Definition 1.1.3. A function f defined on a rectangle $I:=[a,b]$ is said to be of $\Phi-\Lambda-$ bounded variation (that is, $f\in\Phi\Lambda BV([a,b])$) if

$$V_{\Lambda_\Phi}(f,[a,b])=\sup_J\left\{\sum_i\frac{\Phi(|f(I_i|)}{\lambda_i}\right\}<\infty,$$

where Φ is a continuous function defined on $[0,\infty)$ which is strictly increasing from 0 to ∞, $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ is as defined in Definition1.1.2 J is finite collections of non-overlapping subintervals $\{I_i\}$ in $[a,b]$ and $f(I)=f(b)-f(a)$.

Note that, in the above definition, if we substitute

- $\Lambda=\{1\}_{n=1}^{\infty}$, then $\Phi\Lambda BV([a,b])$ reduces to Young's variation $\Phi BV([a,b])$.
- $\Phi(x)=x^p$, $p\geq 1$, then $\Phi\Lambda BV([a,b])$ reduces to Shiba's variation $\Lambda BV^p([a,b])$.

In the above definition of $\Phi-\Lambda-$bounded variation, it is typical to impose a condition that Φ is N function, which we will consider for $\Phi\Lambda BV([a,b])$ from here onwards.

Definition 1.1.4. A function Φ defined on $[0,\infty)$ is said to be N function if following properties are satisfied:

1. $\Phi(0) = 0$,
2. Φ is convex,
3. $\frac{\Phi(x)}{x} \to 0_+$ as $x \to 0$ and $\frac{\Phi(x)}{x} \to \infty$ as $x \to \infty$.

The first condition ensures that a zero-length interval contributes nothing to the variation. The second condition ensures that all the functions in $BV([a,b])$ are in $\Phi\Lambda BV([a,b])$ and the last condition ensures that there are functions in $\Phi\Lambda BV([a,b])$ which are not in $BV([a,b])$. Also, if Φ is N function then in view of [6, p. 225] Φ is strictly increasing, non-negative and continuous function on $[0,\infty)$.

One another condition usually imposed with the space $\Phi - \Lambda-$bounded variation functions, which makes it linear space, is Δ_2 condition [54, p. 273].

Definition 1.1.5. A function Φ is said to satisfy Δ_2 condition if there exists a constant $d \geq 2$ such that $\Phi(2x) \leq d\Phi(x)$, for all $x \geq 0$.

Let

$$c\Phi\Lambda BV([a,b]) = \{f:\ cf \in \Phi\Lambda BV([a,b]) \text{ for some } c>0 \text{ and } f(a)=0\}$$

and

$$||f||_c = \inf\left\{k>0:\ V_{\Lambda_\Phi}\left(\frac{f}{k},[a,b]\right) \leq \frac{1}{\lambda_1}\right\}.$$

Then in view of [34, Theorem 3.3, p. 120], $(c\Phi\Lambda BV([a,b]), ||.||_c)$ is Banach space.

Also, it was observed in [70, Theorem 3, p. 732] that $(\Phi\Lambda BV([a,b]), ||.||_{\Phi_\Lambda})$ is a commutative unital Banach algebra with respect to pointwise operations, where

$$||f||_{\Phi_\Lambda} = ||f||_\infty + V_{\Lambda_\Phi}(f,[a,b]),\ \ f \in \Phi\Lambda BV([a,b]).$$

Schramm and Waterman [55, p. 408] obtained the following result for the order of Fourier coefficient for functions of $\Phi - \Lambda-$ bounded variation.

Theorem C. If $f \in \Phi\Lambda BV([0,2\pi])$ and $n \in \mathbb{N}$, then

$$c_n(f) = O\left(\Phi^{-1}\left(\frac{1}{\sum_{j=1}^{n}\frac{1}{\lambda_j}}\right)\right).$$

In 2011, the following notion of generalized bounded variation [66] was given.

Definition 1.1.6. Let f be a complex valued measurable function defined on $I := [a,b]$; $\{\varphi(n)\}_{n=1}^{\infty}$ is a real sequence such that $\varphi(1) \geq 2$ and $\varphi(n) \uparrow \infty$ as $n \to \infty$; $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a non decreasing sequence of positive numbers such that $\sum_k (\lambda_k)^{-1}$ diverges; and for $1 \leq p \leq \infty, 1 \leq p(n) \uparrow p$ as $n \to \infty$. Then $f \in \Lambda BV(p(n) \uparrow p, \varphi, I)$ if

$$V_{\Lambda_{p(n)}}(f,\varphi,I) = \sup_{n\geq 1}\sup_{\{I_m\}}\left\{V_{\Lambda_{p(n)}}(f,\{I_m\}) : \delta\{I_m\} \geq \frac{b-a}{\varphi(n)}\right\} < \infty,$$

where $\{I_m\}$ is finite collection of non overlapping subintervals of I,

$$V_{\Lambda_{p(n)}}(f,\{I_m\}) = \left(\sum_m \frac{|f(I_m)|^{p(n)}}{\lambda_m}\right)^{1/p(n)},$$

$f(I_m) = f(b_m) - f(a_m)$ and $\delta\{I_m\} = \inf\{|a_m - b_m| : m \in \mathbb{N}\}$.

- If $p(n) = p, \ \forall n \in \mathbb{N}$ then $\Lambda BV(p(n) \uparrow p, \varphi, I)$ coincides with Shiba's variation $\Lambda - BV^p(I)$ [57, p. 8] for $1 \leq p < \infty$.
- If $\varphi(n) = 2^n, \ n = 1, 2, ...,$ and $\Lambda = \{1\}_1^{\infty}$ then the class $\Lambda BV(p(n) \uparrow p, \varphi, I)$ coincides with Kita and Yoneda's variation $BV(p(n) \uparrow p, I)$ [35, Definition 1.1].
- If $\Lambda = \{1\}_1^{\infty}$ then the class $\Lambda BV(p(n) \uparrow p, \varphi, I)$ coincides with Akhobadze's variation $BV(p(n) \uparrow p, I)$ [4, Definition 1, p. 401].

Note that, $f \in \Lambda BV(p(n) \uparrow p, \varphi, I)$ implies f is bounded function on I [66, Lemma 3.1, p. 217]. Also, there exists $f \in BV(p(n) \uparrow p, I)$ having second kind of discontinuities [35, Theorem 3.3, p. 235]. Let $B([0,2\pi])$ be a class of bounded functions in $[0,2\pi]$ then in view of [4, Lemma 2, p. 404], $BV(p(n) \uparrow \infty, \varphi, [0,2\pi])$

and $B([0,2\pi])$ coincides if and only if there is some positive constant K such that for all $n \in \mathbb{N}$, $(\varphi(n))^{\frac{1}{p(n)}} \leq K$.

It was observed [66, Theorem 1, p. 216] that $\Lambda BV(p(n) \uparrow p, \varphi, I)$ is a Banach algebra with respect to pointwise operations and the convolution product with the norm defined for $f \in \Lambda BV(p(n) \uparrow p, \varphi, I)$ as

$$||f||_{var} = ||f||_{\infty} + V_{\Lambda_{p(n)}}(f, \varphi, I).$$

Here, the convolution product [20, cf. p. 50] of two integrable functions f and g, denoted by $f * g$ is given as

$$f * g(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t)g(x-t)dt;\ \forall x \in \overline{\mathbb{T}}.$$

The following inclusion relations were proved by Vyas [67, Lemma 2.7, p. 226] for $1 \leq p < \infty$:

$$BV^p(I) \subset BV(p(n) \uparrow p, \varphi, I)$$

and

$$\bigcup_{1 \leq q < p} BV^q(I) \subset BV(p(n) \uparrow \infty, \varphi, I) \subset BV^p(I).$$

Later on, Goodarzi et al. [39, Corollary 1.7, p. 831] obtained the following inclusion relation for $1 < p \leq \infty$:

$$\bigcup_{1 \leq q < p} \Lambda BV^q([a,b]) \subset \Lambda BV(p(n) \uparrow p, \varphi, [a,b]).$$

The order of Fourier coefficients [66, Theorem 3, p. 216] for the function in $\Lambda BV(p(n) \uparrow \infty, \varphi, [a,b])$ class is as follows.

Theorem D. If $f \in \Lambda BV(p(n) \uparrow \infty, \varphi, [0,2\pi])$, $1 \leq p \leq \infty$ and $n \in \mathbb{Z} \setminus \{0\}$, then

$$c_n(f) = O\left(\frac{1}{\left(\sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right)^{\frac{1}{p(\tau(|n|))}}}\right),$$

where

$$\tau(n) = \min\{k : k \in \mathbb{N}, \varphi(k) \geq n\}, n \geq 1. \tag{1.4}$$

In 2002, Akhobadze [5] further generalized $BV(p(n) \uparrow p, \varphi, I)$ in the following manner:

Definition 1.1.7. Let f be a 2π periodic measurable function and let $p(n)$, p and $\varphi(n)$ be defined as in Definition 1.1.6. Then $f \in B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ if

$$\Lambda(f, p(n) \uparrow p, \varphi, \overline{\mathbb{T}}) = \sup_{m \geq 1} \sup_{h \geq \frac{1}{\varphi(m)}} \left\{ \frac{1}{h} \int_{\overline{\mathbb{T}}} |\, f(x+h) - f(x) \,|^{p(m)} \, dx \right\}^{\frac{1}{p(m)}} < \infty.$$

It was shown [5, Corollary 1, p. 227] that $f \in B\Lambda(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}})$ then f is essentially bounded function. Also, if $f \in BV(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}})$ then $f \in B\Lambda(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}})$ [5, Theorem 3, p. 227]. Further, the following result [5, Theorem 5, p. 228] related to the order of the Fourier coefficient was given.

Theorem E. If $f \in B\Lambda(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}})$ and $n \in \mathbb{Z} \setminus \{0\}$, then

$$c_n(f) = O\left(\frac{1}{|\, n \,|^{\frac{1}{p(\tau(|n|))}}} \right),$$

where $\tau(n)$ is as defined in (1.4).

In the literature, several results related to the order of magnitude of Fourier coefficients have been obtained for functions of generalized bounded variation for various generalized Fourier series.

1.2 Rational Fourier coefficients' properties of one variable functions of generalized bounded variations

The classical Fourier series is generalized to various orthogonal Fourier series to provide flexibility in approximating different functions, better convergence properties, numerical efficiency and applications in other areas of Mathematics, Engineering and Physics. The Fourier series, the Legendre Fourier series, the Chebychev Fourier series and the Walsh Fourier series are suitable for approximating smooth periodic functions with no singularities [31], smooth bounded functions [17], analytic functions [63] and binary functions [26] respectively. Depending on the type of function to be approximated, different types of orthogonal Fourier series provide different convergence properties. The Fourier series, the Legendre Fourier series, the Chebychev Fourier series and the Walsh Fourier series converge uniformly for smooth functions, converge uniformly for square-integrable functions, converge uniformly for analytic functions and converge for periodic continuous functions, respectively. There are many more such orthogonal Fourier series.

The choice of orthogonal Fourier series depends on the specific problem or application and the required accuracy. The classical Fourier series is not very efficient in approximating non-periodic functions and functions with discontinuities or singularities; to overcome such limitations, the rational Fourier series is more suitable [21, 13]. Moreover, with the appropriate selection of parameters, the rational Fourier series exhibits faster convergence and better accuracy than the classical Fourier series for certain functions [46, 50]. It should be noted that the computation of rational Fourier series can be more complex and computationally intensive than Fourier series. Besides theoretical applications [33, 44, 53], the rational Fourier series finds numerous other applications in fields such as control theory [11], system identification [3], signal compression [36], denoising [73], and many more fields [3, 43, 48, 74, 47, 49]. Thus, it is interesting to know the behaviour of the rational Fourier series, which has the orthogonal system as the rational orthogonal system.

The rational orthogonal system is defined $\forall n \in \mathbb{N}$ as

$$\begin{aligned} \phi_0(e^{ix}) &= 1, \\ \phi_n(e^{ix}) &= \frac{\sqrt{1-|\alpha_n|^2}e^{ix}}{1-\overline{\alpha_n}e^{ix}} \prod_{k=1}^{n-1} \frac{e^{ix}-\alpha_k}{1-\overline{\alpha_k}e^{ix}} \\ \text{and } \phi_{-n}(e^{ix}) &= \overline{\phi_n(e^{ix})}, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is complex sequence such that α_k's are in open unit disk $\mathbb{D}$. These α_k's are also called poles of the rational orthogonal system.

In the 1920s, the rational orthogonal system was independently defined by Malmquist [37] and Takenaka [59]. Thus, the rational orthogonal system is also called the Takenaka-Mamquist (or Malmquist-Takenaka) orthogonal system. Achieser [1] observed that the system in (1.5) is complete in $L^2[0, 2\pi]$ if and only if $\sum_{n=1}^{\infty}(1 - |\alpha_k|) = \infty$. The simplest way to satisfy the previous completeness condition is to assume

$$\sup_k |\alpha_k| := r < 1. \tag{1.6}$$

In the sequel, we assume that the condition (1.6) holds.

In 1956, Džrbašyan [19] defined the concept of rational Fourier series with the orthogonal system as a rational orthogonal system. He obtained specific results for the rational Fourier series under condition (1.6).

If f is 2π periodic integerable function, then the rational Fourier series of f is defined as

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\phi_n(e^{ix}), \tag{1.7}$$

where $\hat{f}(n)$ is the n^{th} rational Fourier coefficient of f, given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{\phi_n(e^{ix})}dx. \tag{1.8}$$

If $\alpha_k = 0$, for all $k \in \mathbb{Z}$ in (1.5), the rational Fourier series reduces to Fourier series.

Let $\alpha_k = |\alpha_k|e^{ix_k}$. Then, $\phi_n(e^{ix}) = \rho_n(x)e^{i\theta_n(x)}$ (cf. [19]), where

$$\rho_n(x) = \sqrt{\frac{1 - |\alpha_n|^2}{1 - 2|\alpha_n|\cos(x - x_n) + |\alpha_n|^2}}$$

and

$$\theta_n'(x) = \sum_{k=1}^{n-1} \frac{1 - |\alpha_k|^2}{1 - 2|\alpha_k|\cos(x - x_k) + |\alpha_k|^2} + \frac{1}{2}\left(\frac{1 - |\alpha_n|^2}{1 - 2|\alpha_n|\cos(x - x_n) + |\alpha_n|^2}\right) + \frac{1}{2}.$$

Here, $\theta_n(x)$ is a differentiable and strictly increasing function on $[0, 2\pi]$ [10, cf. p. 465]. It is easy to verify that if all the poles of the rational orthogonal system are zero, then it reduces to the classical exponential system and the rational Fourier series becomes the classical Fourier series. So, the rational Fourier series can be considered, in a way, a generalization of the classical Fourier series.

Note that, there are some properties of the rational Fourier series that differ from the classical Fourier series. It can be seen from the example below that one primary result, related to convolution, fails for rational Fourier coefficients. If $c_n(f)$ and $c_n(g)$ are Fourier coefficients of 2π periodic integrable functions f and g then from [20, cf. p. 51], for $n \in \mathbb{Z}$, it is clear that

$$c_n(f * g) = c_n(f)\ c_n(g).$$

But, the above result does not hold for rational Fourier coefficients as $f(x) = g(x) = \sin x,\ n = 1$ and $\alpha_1 = \frac{1}{2}$ gives $\hat{f}(1) = \hat{g}(1) = \frac{-\sqrt{3}i}{4}$ and $\widehat{f * g}(1) = \frac{-\sqrt{3}}{8}$.

Also, for $n \in \mathbb{Z}$, $|c_n(f)| \leq ||f||_1$ where $||f||_1 = \frac{1}{2\pi}\int_0^{2\pi}|f(x)|dx$. The analogous result for the rational Fourier coefficient slightly differs based on the poles of the rational orthogonal system and is given by

$$|\hat{f}(n)| \leq \sqrt{\frac{1+r}{1-r}}||f||_1,\ n \in \mathbb{Z}.$$

where r is given as in (1.6). Thus, it is interesting to analyse the difference in various properties between rational Fourier series and classical Fourier series.

Tan and Zhou [62] carried out the study of rational Fourier coefficients in 2013. Firstly, they gave an analogous result of Riemann Lebesgue Lemma for rational Fourier series by proving the following inequality [62, Theorem 2.1,p.1739]:

Theorem F. If $f \in L^1([0, 2\pi])$ and $n \in \mathbb{N}$ then

$$|\hat{f}(n)| \leq \frac{1}{4\pi}.\sqrt{\frac{1+r}{1-r}}\omega\left[f, \frac{(1+r)\pi}{(1-r)n}\right] + \frac{||f||_1}{4n}.\frac{r(1+r)^{3/2}}{(1-r)^{7/2}}, \tag{1.9}$$

where $\omega(f;\delta) = \sup_{0<h\leq\delta}||T_h(f) - f||_1$.

Clearly, for $f \in L^1([0, 2\pi])$, using the fact that $\omega(f;\delta) \to 0$ as $\delta \to 0$, it can deduced that $\hat{f}(n) \to 0$ as $n \to \infty$. Thus, it can be considered Riemann Lebesgue Lemma for rational Fourier coefficients. It should be noted that by following proof as in [62, Theorem 2.1, p. 1739] and replacing n by $|n|$, the inequality in (1.9) can be easily proved for $n \in \mathbb{Z} \setminus \{0\}$.

The result of Schramm and Waterman (Theorem C on p. 8) of the order of Fourier coefficients of functions of $\Phi\Lambda BV[0,2\pi]$ is generalized for rational Fourier coefficients [62, Corollar 2.6, p. 1742] as follows:

Theorem G. If $f \in \Phi\Lambda BV([0,2\pi])$ and $n \in \mathbb{N}$, then

$$\hat{f}(n) = O\left(\Phi^{-1}\left(\frac{1}{\sum_{j=1}^{2n} \frac{1}{\lambda_j}}\right)\right).$$

In Chapter 2 of the thesis, the results for order of magnitude of rational Fourier coefficients for functions of $Lip(\alpha,p)([0,2\pi])$ (Definition 1.1.1 on p. 2) class and generalized variations due to Vyas ($\Lambda BV(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$) (Definition 1.1.6 on p. 8) and Akhobadze ($B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$) (Definition 1.1.7 on p. 10) are obtained, using the technique given by Tan and Zhou [62]. Therefore, Theorem A (p. 2), Theorem D (p. 9) and Theorem E (p. 10) have been generalized for the rational Fourier coefficients.

1.3 Order of magnitude of double and multiple Fourier coefficients

The theory initially studied for functions having one variable is often extended for functions with two variables. The properties of these two variable functions are compared with the properties of single variable functions.

The double Fourier series of an integrable function f, which is 2π periodic in both variables, is defined as

$$f(x,y) \sim \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{mn}(f) e^{imx} e^{iny}, \tag{1.10}$$

where $c_n(f)$ is the n^{th} Fourier coefficient of f, given by

$$c_{mn}(f) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y)e^{-imx}e^{-iny}dxdy. \tag{1.11}$$

The Riemann Lebesgue Lemma is extended for the double Fourier series as below:

If $f \in L^1(\overline{\mathbb{T}}^2),\ (m,n) \in \mathbb{Z}^2$ then $c_{mn}(f) \to 0$ as $|mn| = \sqrt{|m|^2+|n|^2} \to \infty$.

This Riemann Lebesgue Lemma does not estimate the order of double Fourier coefficients and thus, various subclasses of $L^1(\overline{\mathbb{T}}^2)$ are studied. One such subclass is $Lip(p;\zeta,\beta)(\overline{\mathbb{T}}^2)$ class, which is extension of $Lip(\beta,p)(\overline{\mathbb{T}})$ for two variable functions. Vyas and Darji [71, Definition 1.1, p. 27]defined this class as below:

Definition 1.3.1. Let $f \in L^p(\overline{\mathbb{T}}^2), p \geq 1$ and $\zeta, \beta \in (0,1]$, we say that $f \in Lip(p;\zeta,\beta)(\overline{\mathbb{T}}^2)$, if $\omega^{(p)}(f;\delta,\gamma) = O(\delta^\zeta\gamma^\beta)$ as δ and $\gamma \to 0$, where

$$\omega^{(p)}(f;\delta,\gamma) = \sup\left\{\left(\frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2}|\Delta f(x,y;h,k)|^p dxdy\right)^{\frac{1}{p}}; 0 < h \leq \delta, 0 < k \leq \gamma\right\}$$

and $\Delta f(x,y;h,k) = f(x+h,y+k) - f(x+h,y) - f(x,y+k) + f(x,y)$.

The class $Lip(p;\zeta,\beta)(\overline{\mathbb{T}}^2)$ reduces to $Lip(\zeta,\beta)(\overline{\mathbb{T}}^2)$ for $p = \infty$.

Vyas and Darji [71] obtained the order of double Fourier coefficients of functions in $Lip(p;\zeta,\beta)(\overline{\mathbb{T}}^2)$ class.

Theorem H. If $f \in Lip(p;\zeta,\beta)(\overline{\mathbb{T}}^2)p \geq 1, \zeta, \beta \in (0,1]$ and $(m,n) \in (\mathbb{Z}\setminus\{0\})^2$ then

$$c_{mn}(f) = O\left(\frac{1}{|m|^\zeta|n|^\beta}\right).$$

Soon after Jordan studied bounded variation, several mathematicians began to study the notion of bounded variation for functions of two variables. When defining bounded variation functions for two variables, one has different choices depending on the specific context and requirements of the given problem, as the functions of two variables can exhibit more complex behaviour compared to single variable functions. The notion of bounded variation is extended for two variables in various ways, like in the sense of Vitali, Fréchet, Hardy, Arzelà and others as can be seen in [14]. Here, we will consider definitions of functions of generalized bounded variation mainly in the sense of Vitali and Hardy. In this section, the following notation for two variable functions f will be frequently used,

$$f(I \times J) := f([a,b] \times [c,d]) = f(b,d) - f(a,d) - f(b,c) + f(a,c)$$

The definition for two variable functions of bounded variation in the sense of Vitali [14, Definition V, p. 825] is as follows:

Definition 1.3.2. A function f defined on a rectangle $R^2 := [a,b] \times [c,d]$ is said to be of bounded variation in the sense of Vitali (written as, $f \in BV_V(R^2)$) if

$$V(f, R^2) = \sup_{J_1, J_2} \left\{ \sum_i \sum_j |f(I_i \times K_j)| \right\} < \infty,$$

where J_1 and J_2 are finite collections of non-overlapping subintervals $\{I_i\}$ and $\{K_j\}$ in $[a,b]$ and $[c,d]$ respectively.

It can easily be verified that the function of bounded variation in the sense of Vitali need not be bounded from the example [25, Example 1.19(i), p. 23] below:

Example 1.3.1. Let $f : [0,1]^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} \frac{1}{x} + \frac{1}{y}, & \text{if } x \neq 0 \text{ and } y \neq 0, \\ \frac{1}{x}, & \text{if } x \neq 0 \text{ and } y = 0, \\ \frac{1}{y}, & \text{if } x = 0 \text{ and } y \neq 0, \\ 0, & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Clearly, $V(f, [0,1]^2) = 0$. Thus, $f \in BV_V([0,1]^2)$ but f is unbounded on $[0,1]^2$. This is quite different from its one-dimensional analogous definition, where if a function is of bounded variation on some interval, then it is always bounded on that interval. We need a definition of bounded variation for two variable functions that capture the overall variation in a manner analogous to the concept of bounded variation for single variable functions by separately analysing variations along each coordinate direction. This can be achieved by considering bounded variation in the sense of Hardy [14, Defintion H, p. 825] for two variable functions defined as follows.

Definition 1.3.3. If $f \in BV_V(R^2 := [a,b] \times [c,d])$ is such that the marginal functions $f(.,c) \in BV([a,b])$ and $f(a,.) \in BV([c,d])$ then f is said to be of bounded variation in the sense of Hardy (written as is, $f \in BV_H(R^2)$).

It would follow that if $f \in BV_H(R^2)$ then f is bounded on R^2 because

$$\begin{aligned}|f(x,y)| \leq & |f(x,y) - f(a,y) - f(x,c) + f(a,c)| \\ & + |f(a,y) - f(a,c)| + |f(x,c) - f(a,c)| + |f(a,c)| \\ \leq & V(f, R^2) + V(f(a,.), [c,d]) + V(f(.,c), [a,b]) + |f(a,c)|.\end{aligned}$$

It is also known [2, p. 722] that if $f \in BV_H(R^2)$, then the discontinuities of f are located on a countable number of lines parallels to some of the axes. Hence, f is Lebesgue measurable over R^2. Subsequently, f is also bounded over R^2 and hence, $f \in L^1(R^2)$.

The upcoming three results for double Fourier coefficients are due to Fülöp and Móricz [22, Case n=2]. Later, Schramm and Waterman [56] proved these results using different techniques.

Theorem I. If $f \in BV_V(\overline{\mathbb{T}}^2) \cap L^1(\mathbb{T}^2)$ and $(m,n) \in (\mathbb{Z} \setminus \{0\})^2$ then

$$c_{mn}(f) = O\left(\frac{1}{|mn|}\right).$$

Corollary A. If $f \in BV_H(\overline{\mathbb{T}}^2)$ and $(m,n) \in (\mathbb{Z} \setminus \{0\})^2$ then

$$c_{mn}(f) = O\left(\frac{1}{|mn|}\right).$$

Corollary B. If $f \in BV_H(\overline{\mathbb{T}}^2)$ and $n \in \mathbb{Z} \setminus \{0\}$ then

$$c_{0n}(f) = O\left(\frac{1}{|n|}\right).$$

The concepts of bounded variation in the sense of Vitali and Hardy are further generalized. Vyas and Darji [71, Definition 1.2, p. 28] defined $p-\Lambda-$bounded variation in the sense of Vitali and Hardy as follows:

Definition 1.3.4. Given $\Lambda = (\Lambda_1, \Lambda_2)$ where $\Lambda_k = \{\lambda_{(k,n)}\}_{n=1}^{\infty}$ and $\{\lambda_{(k,n)}\}_{n=1}^{\infty}$

is a non-decreasing sequence of positive numbers such that $\sum_n \frac{1}{\lambda_{(k,n)}}$ diverges for $k = 1,\ 2$, and $p \geq 1$, a measurable function f defined on a rectangle R^2 is said to be of $p-\Lambda-$bounded variation (that is, $f \in \Lambda BV^p(R^2)$) if

$$V_{\Lambda_p}(f, R^2) = \sup_{J_1, J_2} \left\{ \left(\sum_i \sum_j \frac{|f(I_i \times K_j|^p}{\lambda_{(1,i)}\lambda_{(2,j)}} \right)^{\frac{1}{p}} \right\} < \infty,$$

where R^2, J_1, J_2, I_i and K_j are as defined earlier in the Definition 1.3.2.

Let f be as defined earlier in the Example 1.3.1 on p. 16 and thus $V_{\Lambda_p}(f, [0,1]^2) = 0$. Thus, a function $f \in \Lambda BV^p(R^2)$ need not be bounded.

This class is further defined in the sense of Hardy as follows.

Definition 1.3.5. If $f \in \Lambda BV^p(R^2)$ is such that the marginal functions $f(.,c) \in \Lambda_1 BV^p([a,b])$ (see Definition 1.1.2 on p. 3) and $f(a,.) \in \Lambda_2 BV^p([c,d])$ then f is said to be of $p-\Lambda^*-$bounded variation (that is, $f \in \Lambda^* BV^p(R^2)$).

If $f \in \Lambda^* BV^p(R^2)$ then f is bounded, as

$$\begin{aligned} |f(x,y)| \leq & |f(x,y) - f(a,y) - f(x,c) + f(a,c)| \\ & + |f(a,y) - f(a,c)| + |f(x,c) - f(a,c)| + |f(a,c)| \\ \leq & (\lambda_{(1,1)}\lambda_{(1,2)})^{\frac{1}{p}} V_{\Lambda_p}(f, R^2) + \lambda_{(1,2)}^{\frac{1}{p}} V_{\Lambda_{2p}}(f(a,.), [c,d]) \\ & + \lambda_{(1,1)}^{\frac{1}{p}} V_{\Lambda_{1p}}(f(.,c), [a,b]) + |f(a,c)|. \end{aligned}$$

Note that if we substitute

- $\Lambda_1 = \Lambda_2 = \{1\}$ and $p = 1$, then the classes $\Lambda BV^p(R^2)$ and $\Lambda^* BV^p(R^2)$ reduce to the classes $BV_V(R^2)$ and $BV_H(R^2)$ respectively.
- $p = 1$, then the classes $\Lambda BV^p(R^2)$ and $\Lambda^* BV^p(R^2)$ reduce to the classes $\Lambda BV(R^2)$ (defined by Sablin [52] and Saakyan [51]) and $\Lambda^* BV(R^2)$ respectively.
- $\Lambda_1 = \Lambda_2 = \{1\}$, then the classes $\Lambda BV^p(R^2)$ and $\Lambda^* BV^p(R^2)$ reduce to the classes $BV_V^p(R^2)$ (defined by Golubov [27]) and $BV_H^p(R^2)$ respectively.

For any $\Lambda = (\Lambda_1, \Lambda_2), I_i, K_j$ and $p \geq 1$ as defined in Definition 1.12, we have

$$\left(\sum_i \sum_j \frac{|f(I_i \times K_j)|^p}{\lambda_{(1,i)}\ \lambda_{(2,j)}}\right)^{\frac{1}{p}} \leq \left(\frac{1}{\lambda_{(1,1)}\ \lambda_{(2,1)}}\right)^{\frac{1}{p}} \left(\sum_i \sum_j |f(I_i \times K_j)|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{\lambda_{(1,1)}\ \lambda_{(2,1)}}\right)^{\frac{1}{p}} \sum_i \sum_j |f(I_i \times K_j)|.$$

This implies

$$BV_V(R^2) \subset BV_V^{(p)}(R^2) \subset \Lambda BV^{(p)}(R^2)$$

and hence

$$BV_H(R^2) \subset BV_H^{(p)}(R^2) \subset \Lambda^* BV^{(p)}(R^2).$$

Dyachenko and Waterman [18, Proposition 1, p.401] showed that class $\Lambda^* BV(R^2)$ contains everywhere discontinuous function.

Vyas and Darji [15, Definition 3.1, p. 184] gave a definition of $p - \Lambda$ variation for a two variable function $f : \sigma \to \mathbb{B}$ denoted by $\Lambda BV^p(\sigma, \mathbb{B})$, where σ is any non-empty compact sub-rectangle of $\mathbb{R}^2$ and $\mathbb{B}$ is a Banach algebra. In view of [15, Theorem 3.6, p. 185], for a commutative Banach algebra $\mathbb{B}$, $\Lambda BV^p(\sigma, \mathbb{B})$ with suitable variation norm is a commutative Banach algebra with respect to pointwise operations.

Vyas and Darji [71, Theorem 2.1 and Corollary 2.1, p. 30] obtained the following two results related to the order of double Fourier coefficients

Theorem J. If $f \in \Lambda BV^{(p)}(\overline{\mathbb{T}}^2) \cap L^p(\overline{\mathbb{T}}^2)$ $(p \geq 1)$ and $(m, n) \in (\mathbb{Z} \setminus \{0\})^2$, then

$$c_{mn}(f) = O\left(\frac{1}{\left(\sum_{j=1}^{|m|} \sum_{k=1}^{|n|} \frac{1}{\lambda_{(1,j)}\ \lambda_{(2,k)}}\right)^{\frac{1}{p}}}\right). \tag{1.12}$$

Corollary C. If $f \in \Lambda^* BV^{(p)}(\overline{\mathbb{T}}^2) \cap L^p(\overline{\mathbb{T}}^2)$ $(p \geq 1)$ and $(m, n) \in (\mathbb{Z} \setminus \{0\})^2$, then equation (1.12) holds.

The variations $\Phi\Lambda BV$ and $\Phi\Lambda^* BV$ are defined [72, Definition 1, p. 1153] as follows:

Definition 1.3.6. A measurable function f defined on a rectangle R^2 is said to be of $\Phi - \Lambda-$ bounded variation (that is, $f \in \Phi\Lambda BV(R^2)$) if

$$V_{\Lambda_\Phi}(f, R^2) = \sup_{J_1, J_2}\left\{\sum_i \sum_j \frac{\Phi(|f(I_i \times K_j|)}{\lambda_{(1,i)}\lambda_{(2,j)}}\right\} < \infty$$

where Φ is an continuous function (see Definition 1.1.4) defined on $[0, \infty)$ which is strictly increasing from 0 to ∞ and R^2, J_1, J_2, I_i and K_j are as defined earlier in the Definition 1.3.2.

The above generalized variation is in the sense of Vitali. Here, $f \in \Phi\Lambda BV(R^2)$ need not be bounded as for the unbounded function f as defined earlier in the Example 1.3.1 on p. 16, we have $V_{\Lambda_\Phi}(f, [0,1]^2) = 0$.

$\Phi - \Lambda-$bounded variation is given in the sense of Hardy as follows.

Definition 1.3.7. If $f \in \Phi\Lambda BV(R^2)$ is such that the marginal functions $f(., c) \in \Phi\Lambda_1 BV([a, b])$ and $f(a, .) \in \Phi\Lambda_2 BV([c, d])$(see Definition 1.1.3 on p. 6) then f is said to be of $\Phi - \Lambda^*-$ bounded variation (that is, $f \in \Phi\Lambda^* BV(R^2)$).

Note that in view of [72, Corollary 1, p. 1156] if $f \in \Phi\Lambda^* BV(R^2)$ then f is bounded function in R^2. Note that, if we substitute $\Phi(x) = x^p$, $p \geq 1$, then $\Phi\Lambda BV([a, b])$ and $\Phi\Lambda^* BV([a, b])$ reduce to $\Lambda BV^p([a, b])$ and $\Lambda^* BV^p([a, b])$ respectively. In the above definition of $\Phi - \Lambda-$bounded variation and $\Phi - \Lambda^*-$ bounded variation, it is typical to impose a condition that Φ is N function (see Definition 1.1.4), which we will assume for $\Phi\Lambda BV(R^2)$ and $\Phi\Lambda BV(R^2)$ from here onwards.

Vyas and Darji [72, Theorem 1, Corollary 1 and Corollary 2] obtained the following three results related to the order of double Fourier coefficients for functions of $\Phi - \Lambda-$bounded variation both in the sense of Vitali and Hardy.

Theorem K. If Φ satisfies Δ_2 condition, $f \in \Phi\Lambda BV(\overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2)$ and $(m, n) \in (\mathbb{Z} \setminus \{0\})^2$ then

$$c_{mn}(f) = O\left(\Phi^{-1}\left(\frac{1}{\sum_{j=1}^{|m|}\sum_{k=1}^{|n|}\frac{1}{\lambda_{(1,j)}\lambda_{(2,k)}}}\right)\right). \tag{1.13}$$

Corollary D. If Φ satisfies Δ_2 condition and $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^2)$ and $(m, n) \in (\mathbb{Z} \setminus \{0\})$ then condition (1.13) holds.

Corollary E. If Φ satisfies Δ_2 condition, $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^2)$ and $m \in \mathbb{Z} \setminus \{0\}$ then

$$c_{m0}(f) = O\left(\Phi^{-1}\left(\frac{1}{\sum_{j=1}^{|m|} \frac{1}{\lambda_{(1,j)}}}\right)\right).$$

By extending the class of generalized bounded variation due to Schramm and Waterman [55] for two-variable functions and Definition 1.3.6, Darji and Vyas [16, p. 2] gave the following two definitions of $(\Phi, \Psi)(\Lambda, \Gamma)BV$ and $(\Phi, \Psi)(\Lambda, \Gamma)^* BV$.

Definition 1.3.8. Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $\Gamma = \{\gamma_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive numbers such that $\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k = \lim_{n\to\infty} \sum_{k=1}^{n} \gamma_k = \infty$, let Φ and Ψ be continuous and increasing functions on $[0, \infty)$ and J_1, J_2, I_i and K_j are as defined earlier in the Definition 1.3.2. Then a complex measurable function $f \in \overline{\mathbb{T}}^2$ is said to be of $(\Phi, \Psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ if

$$V_{(\Lambda,\Gamma)_{(\Psi,\Phi)}}(f, \overline{\mathbb{T}}^2) = \sup_{J_1, J_2} \left(\sum_k \frac{1}{\gamma_k} \Psi\left(\sum_i \frac{\Phi(|f(I_i \times K_j)|)}{\lambda_i}\right)\right) < \infty.$$

The above variation $(\Phi, \Psi)(\Lambda, \Gamma)BV$ is in Vitali sense. Note that, a function $f \in (\Phi, \Psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ need not be a bounded function.

Clearly, $(\Phi, \Psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ reduces to $\Phi\Lambda BV(\overline{\mathbb{T}}^2)$ for $\Psi(x) = x$.

In the above definition of $(\Phi, \Psi)(\Lambda, \Gamma)$ bounded variation, it is typical to impose a condition that Φ and Ψ are N function (see Definition 1.1.4), which we will assume for $(\Phi, \Psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ from here onwards.

The variation $(\Phi, \Psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ in Hardy sense is given as follow.

Definition 1.3.9. If $f \in (\Phi, \Psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2)$ and the marginal functions $f(0, .) \in \Phi\Lambda BV(\overline{\mathbb{T}})$ (see Definition 1.1.3) and $f(., 0) \in \Psi\Gamma BV(\overline{\mathbb{T}})$, then f is said to belong to $(\Phi, \Psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$.

If $f \in (\Phi, \Psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$, then f is bounded on $\overline{\mathbb{T}}^2$ [16, Lemma 2.2, p. 5].

In 2020, Darji and Vyas [16, Theorem 2.1, Corollary 2.3 and Corollary 2.5] obtained the following three results related to the order of double Fourier coefficients.

Theorem L. If $f \in (\Phi, \Psi)(\Lambda, \Gamma)BV(\overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2)$ and $(m, n) \in (\mathbb{Z} \setminus \{0\})^2$, then

$$c_{mn}(f) = O\left(\Phi^{-1}\left(\frac{1}{\Lambda_{|m|}}\Psi^{-1}\left(\frac{1}{\Gamma_{|n|}}\right)\right)\right), \tag{1.14}$$

where $\Lambda_{|m|} = \sum_{j=1}^{|m|} \lambda_j^{-1}$ and $\Gamma_{|n|} = \sum_{k=1}^{|n|} \gamma_k^{-1}$.

Corollary F. If $f \in (\Phi, \Psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$ and $(m, n) \in (\mathbb{Z}\setminus\{0\})^2$, then condition (1.14) holds.

Corollary G. If Φ, Ψ satisfies Δ_2 condition, $f \in (\Phi, \Psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$ and $(m, 0) \in (\mathbb{Z} \setminus \{0\})^2$, then

$$c_{m0}(f) = O\left(\Phi^{-1}\left(\frac{1}{\Lambda_{|m|}}\right)\right),$$

where $\Lambda_{|m|} = \sum_{j=1}^{|m|} \lambda_j^{-1}$.

Vyas [69, Definition 2.1, p. 435] extended the definition $\Lambda BV(p(n) \uparrow p, \varphi, I)$ (see Definition 1.1.6) of one variable for two variable as follows.

Definition 1.3.10. Let f be a complex valued measurable function defined on $R^2 := I^{(1)} \times I^{(2)} := [a_1, b_1] \times [a_2, b_2]$; $\Lambda = \left(\Lambda^{(1)}, \Lambda^{(2)}\right)$, where $\Lambda^{(t)} = \left\{\lambda_k^{(t)}\right\}_{k=1}^{\infty}$ is a non decreasing sequence of positive numbers such that $\sum_k \left(\lambda_k^{(t)}\right)^{-1}$ diverges for $t = 1, 2$; $\{\varphi(n)\}_{n=1}^{\infty}$ is a real sequence such that $\varphi(1) \geq 2$ and $\varphi(n) \uparrow \infty$ as $n \to \infty$ and for $1 \leq p \leq \infty, 1 \leq p(n) \uparrow p$ as $n \to \infty$. Then $f \in \Lambda BV(p(n) \uparrow p, \varphi, R^2)$ if

$$V_{\Lambda_{p(n)}}(f, \varphi(n), R^2) = \sup_{n \geq 1} \sup_{\left\{I_i^{(1)} \times I_j^{(2)}\right\}} \left\{ V_{\Lambda_{p(n)}}\left(f, \left\{I_i^{(1)} \times I_j^{(2)}\right\}\right) \right.$$
$$\left. : \delta\left\{I_i^{(1)} \times I_j^{(2)}\right\} \geq \frac{(b_1 - a_1)(b_2 - a_2)}{\varphi(n)^2} \right\} < \infty,$$

where $\left\{I_i^{(1)}\right\}$ and $\left\{I_j^{(2)}\right\}$ are finite collections of non overlapping subintervals of

$I^{(1)}$ and $I^{(2)}$ respectively,

$$V_{\Lambda_{p(n)}}\left(f,\left\{I_i^{(1)}\times I_j^{(2)}\right\}\right)=\left(\sum_i\sum_j\frac{\left|f\left(I_i^{(1)}\times I_j^{(2)}\right)\right|^{p(n)}}{\lambda_i^{(1)}\lambda_j^{(2)}}\right)^{1/p(n)},$$

$$f\left(I^{(1)}\times I^{(2)}\right)=f(b_1,b_2)-f(a_1,b_2)-f(b_1,a_2)+f(a_1,a_2)$$

and

$$\delta\left\{I_i^{(1)}\times I_j^{(2)}\right\}:=\delta\{[s_{i-1},s_i]\times[t_{j-1},t_j]\}=\inf_{i,j}|(s_i-s_{i-1})(t_j-t_{j-1})|.$$

The above variation is in the sense of Vitali. Note that, a function $f\in\Lambda BV(p(n)\uparrow p,\varphi,R^2)$ need not be bounded (see [69, Example 2.2, p.436]). The above variation in the sense of Hardy is given below.

Definition 1.3.11. If $f\in\Lambda BV(p(n)\uparrow p,\varphi,R^2)$ and the marginal functions $f(.,a_2)\in\Lambda^{(1)}BV(p(n)\uparrow p,\varphi,[a_1,b_1])$ and $f(a_1,.)\in\Lambda^{(2)}BV(p(n)\uparrow p,\varphi,[a_2,b_2])$, then $f\in\Lambda^*BV(p(n)\uparrow p,\varphi,R^2)$.

Note that, the function $f\in\Lambda^*BV(p(n)\uparrow p,\varphi,R^2)$ is bounded. If we substitute $\Lambda=(\{1\},\{1\})$ in $\Lambda BV(p(n)\uparrow p,\varphi,R^2)$ and $\Lambda^*BV(p(n)\uparrow p,\varphi,R^2)$ then we get $BV_V(p(n)\uparrow p,\varphi,R^2)$ and $BV_H(p(n)\uparrow p,\varphi,R^2)$ (see Definitions due to Vyas [68, Definition 2.1 and 2.2]).

Let $B(R^2)$ represent a class of two variable bounded functions on $R^2:=[a,b]\times[c,d]$ then in view of [68, Lemma 3.3, p. 150], for $1\le p<\infty$, we have

$$B(R^2)\cap BV_V^p(R^2)\subseteq B(R^2)\cap BV_V(p(n)\uparrow\infty,\varphi,R^2),$$

$$BV_H^p(R^2)\subseteq BV_H(p(n)\uparrow\infty,\varphi,R^2),$$

$$B(R^2)\cap\bigcup_{1\le q<p}BV_V^q(R^2)\subseteq B(R^2)\cap BV_V(p(n)\uparrow p,\varphi,R^2)\subseteq B(R^2)\cap BV_V^p(R^2)$$

and

$$\bigcup_{1\le q<p}BV_H^q(R^2)\subseteq BV_H(p(n)\uparrow p,\varphi,R^2)\subseteq BV_H^p(R^2)$$

In 2015, Vyas [69, Theorem 3.1, Corollary 3.3 and Theorem 3.4] obtained the following three results related to the order of double Fourier coefficients.

Theorem M. If $f \in \Lambda BV(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2) \cap L^\infty(\overline{\mathbb{T}}^2)$ and $(m,n) \in (\mathbb{Z} \setminus \{0\})^2$, then

$$c_{mn}(f) = O\left(\frac{1}{\left(\sum_{j=1}^{|m|} \sum_{k=1}^{|n|} \frac{1}{\lambda_j^{(1)} \lambda_k^{(2)}} \right)^{\frac{1}{p(\tau(|mn|))}}} \right), \tag{1.15}$$

where

$$\tau(n) = \min\{k : k \in \mathbb{N}, \varphi(k) \geq n\}, n \geq 2. \tag{1.16}$$

Corollary H. If $f \in \Lambda^* BV(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$ and $(m,n) \in (\mathbb{Z} \setminus \{0\})^2$, then condition (1.15) holds.

Corollary I. If $f \in \Lambda^* BV(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$ and $m \in \mathbb{Z} \setminus \{0\}$, then

$$c_{m0}(f) = O\left(\frac{1}{\left(\sum_{j=1}^{|m|} \frac{1}{\lambda_j^{(1)}} \right)^{\frac{1}{p(\tau(|m|))}}} \right),$$

where $\tau(|m|)$ is as defined in (1.16).

In the rest of this section, we will consider theory for several variables, extending definitions and results for two variable functions to N variable functions for $N \in \mathbb{N}$ and $N > 2$.

For a function, $f \in L^1(\overline{\mathbb{T}}^N)$, which is 2π periodic in all the variables, multiple Fourier series of f is defined as

$$f(x_1, x_2, ..., x_N) \sim \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} ... \sum_{k_N \in \mathbb{Z}} c_{k_1 k_2 ... k_N}(f) e^{ik_1 x_1} e^{ik_2 x_2} ... e^{ik_N x_N},$$

where $N \in \mathbb{N}$ and $c_{k_1 ... k_N}(f)$ is the $(k_1, ..., k_N)^{th}$ multiple Fourier coefficient of f given by

$$c_{k_1 ... k_N}(f) = \frac{1}{(2\pi)^N} \int ... \int_{\overline{\mathbb{T}}^N} f(x_1, ..., x_n) e^{-ik_1 x_1} ... e^{-ik_N x_N} dx_1 ... dx_N.$$

The following Riemann Lebesgue Lemma holds for multiple Fourier coefficients: If $f \in L^1(\overline{\mathbb{T}}^N), (k_1, ..., k_N) \in \mathbb{Z}^N$, then $c_{k_1 ... k_N} \to 0$ as $\sqrt{|k_1|^2 + ... + |k_N|^2} \to \infty$.

Now, since Riemann Lebesgue Lemma does not give the order of magnitude of multiple Fourier coefficients, we will look into various subclasses of $L^1(\overline{\mathbb{T}}^N)$, namely $Lip(p;\beta_1,...,\beta_N)(\overline{\mathbb{T}}^N)$ and class of functions of generalized bounded variation.

Vyas and Darji [71, p. 33] gave the following definition.

Definition 1.3.12. Given $(x_1,...,x_N)\in\overline{\mathbb{T}}^N$ and $f\in L^p(\overline{\mathbb{T}}^N)$, where $p\geq 1$, the p-integral modulus of continuity of f is defined as

$$\omega^{(p)}(f;\delta_1,...,\delta_N)$$
$$=\sup\left\{\left(\frac{1}{(2\pi)^N}\int...\int_{\overline{\mathbb{T}}^N}|\Delta f(x_1,...,x_N;h_1,...,h_N)|^p dx_1...dx_N\right)^{\frac{1}{p}}\right.$$
$$\left.:0<h_i\leq\delta_i\ \ \forall i=1,2,...,N\right\},$$

where

$$\Delta f(x_1,...,x_N;h_1,...,h_N)=\sum_{u_1=0}^{1}...\sum_{u_N=0}^{1}(-1)^{u_1+...+u_N}f(x_1+u_1h_1,...,x_N+u_Nh_N).$$

For $p\geq 1$ and $\beta_i\in(0,1]$, for all $i=1,2,..N$, we say that $f\in Lip(p;\beta_1,...,\beta_N)(\overline{\mathbb{T}}^N)$ if

$$\omega^{(p)}(f;\delta_1,...,\delta_N)=O(\delta_1^{\beta_1}...\delta_N^{\beta_N}).$$

If $p=\infty$ then $Lip(p;\beta_1,...,\beta_N)(\overline{\mathbb{T}}^N)$ reduces to $Lip(\beta_1,...,\beta_N)(\overline{\mathbb{T}}^N)$.

Vyas and Darji [71, Theorem 4.3, p. 35] obtained the following result for the order of multiple Fourier coefficients for functions in $Lip(p;\beta_1,...,\beta_N)(\overline{\mathbb{T}}^N)$ class.

Theorem N. If $Lip(p;\beta_1,...,\beta_N)(\overline{\mathbb{T}}^N)$, $p\geq 1$, $\beta_1,...,\beta_N\in(0,1]$ and $(k_1,...,k_N)\in(\mathbb{Z}\setminus\{0\})^N$ then

$$c_{k_1...k_N}(f)=O\left(\frac{1}{|k_1|^{\alpha_1}...|k_N|^{\alpha_N}}\right).$$

Vyas and Darji [72, Definition 1, p. 1153] defined $\Phi-\Lambda-$ bounded variation in the sense of Vitali and Hardy.

Definition 1.3.13. A measurable function f defined on a rectangle $R^N:=[a_1,b_1]\times[a_2,b_2]\times...\times[a_N,b_N]$ is said to be of $\Phi-\Lambda-$ bounded variation (that is,

$f \in \Phi\Lambda BV(R^N))$ if

$$V_{\Lambda_\Phi}(f, R^N) = \sup_{J_1,...,J_N} \left\{ \sum_{k_1} ... \sum_{k_N} \frac{\Phi(|f(I_{(1,k_1)} \times ... \times I_{(N,k_N)})|)}{\lambda_{(1,k_1)}...\lambda_{(N,k_N)}} \right\} < \infty$$

where Φ is a N function (see Definition 1.1.4) defined on $[0, \infty)$ which is strictly increasing from 0 to ∞; $\Lambda = (\Lambda_1, \Lambda_2, ..., \Lambda_N)$ where $\Lambda_k = \{\lambda_{(k,n)}\}_{n=1}^{\infty}$ and $\{\lambda_{(k,n)}\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive numbers such that $\sum_n \frac{1}{\lambda_{(k,n)}}$ diverges for $k = 1, 2, ..., N$; $J_1, ..., J_{N-1}$ and J_N are finite collections of non-overlapping subintervals $\{I_{(1,k_1)}\}, ..., \{I_{(N-1,k_{N-1})}\}$ and $\{I_{(N,k_N)}\}$ in $[a_1, b_1], ..., [a_{N-1}, b_{N-1}]$ and $[a_N, b_N]$ respectively; and $f(J_1 \times ... \times J_N) = f(J_1 \times ... \times J_{N-1}, b_N) - f(J_1 \times ... \times J_{N-1}, a_N)$, here $f(J_1) = f(b_1) - f(a_1)$, $f(J_1 \times J_2) = f(b_1, b_2) + f(a_1, a_2) - f(b_1, a_2) - f(a_1, b_2)$ and so on.

The above generalized variation is in the sense of Vitali. Here, $f \in \Phi\Lambda BV(R^N)$ need not be bounded. This class is further generalized in the sense of Hardy as follows.

Definition 1.3.14. A function $f \in \Phi\Lambda BV(R^N)$ is said to be of $\Phi - \Lambda^* -$ bounded variation (that is, $f \in \Phi\Lambda^* BV(R^N)$) if for each of its marginal functions

$$f(x_1, ..., x_{i-1}, a_i, x_{i+1}, ..., x_N) \in \Phi(\Lambda_1, ..., \Lambda_{i-1}, \Lambda_{i+1}, ..., \Lambda_N)^* BV^{(p)}(R^N(a_i)),$$

$\forall i = 1, 2, ..., N$, where

$$R^N(a_i) = \{(x_1, ..., x_{i-1}, x_{i+1}, ..., x_N) \in \mathbb{R}^{N-1} : x_k \in [a_k, b_k] \\ \forall k = 1, ..., i-1, i+1, ...N\}.$$

Note that $f \in \Phi\Lambda^* BV(R^N)$ implies that f is bounded function in R^2.

Vyas and Darji [72, Theorem 2, Corollary 3 and 4] obtained the following three results for the order of multiple Fourier coefficients.

Theorem O. If $f \in \Phi\Lambda BV(\overline{\mathbb{T}}^N) \cap L^1(\overline{\mathbb{T}}^N)$ and $(k_1, ..., k_N) \in (\mathbb{Z} \setminus \{0\})^N$ then

$$c_{k_1...k_N} = O\left(\Phi^{-1}\left(\frac{1}{\sum_{i_1=1}^{|k_1|} \cdots \sum_{i_N=1}^{|k_N|} \frac{1}{\lambda_{(1,i_1)}...\lambda_{(N,i_N)}}}\right)\right). \qquad (1.17)$$

Corollary J. If $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^N)$ and $(k_1, ..., k_N) \in (\mathbb{Z} \setminus \{0\})^N$ then condition (1.17) holds.

Corollary K. If Φ satisfies Δ_2 condition, $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^N)$ and $(k_1, ..., k_N) \in \mathbb{Z}^N$ is such that $k_j \neq 0$ for $(1 \leq) j_1 < ... < j_M (\leq N)$ and $k_j = 0$ for $(1 \leq) l_1 < ... < l_{N-M} (\leq N)$, where $\{l_1, ..., l_{N-M}\}$ is the complementary set of $\{j_1, ..., j_M\}$ with respect to $\{1, ..., N\}$, then

$$c_{k_1 \ldots k_N} = O\left(\Phi^{-1}\left(\frac{1}{\sum_{i_1=1}^{|k_{j_1}|} \cdots \sum_{i_M=1}^{|k_{j_M}|} \frac{1}{\lambda_{(j_1,i_1)} \cdots \lambda_{(j_M,i_M)}}}\right)\right).$$

In Chapter 3 of the thesis, the Riemann Lebesgue Lemma for double rational Fourier coefficients is obtained. The orders of magnitude of double rational Fourier coefficients for $Lip(p; \zeta, \beta)(\overline{\mathbb{T}}^2)$ class (Definition 1.3.1 on p. 15) and the classes of generalized bounded variation functions like $\Phi\Lambda^* BV(\overline{\mathbb{T}}^2)$ (Definition 1.3.7 on p. 20), $(\Phi, \Psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$ (Definition 1.3.9 on p. 21) and $\Lambda^* BV(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2)$ (Definition 1.3.11 on p. 1.3.11) are estimated. Also, the Akhobadze class of variation, $B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ (Definition 1.1.7 on p. 10), is extended for two variable function and the order of double rational Fourier coefficients for functions from this extended class is obtained. Furthermore, these results of the order of double rational Fourier coefficients are extended for multiple rational Fourier coefficients.

1.4 Rate of convergence of Fourier and double Fourier series

The significant aspect of studying the Fourier series of an integrable function is determining the conditions of convergence of the Fourier series, that is, the general property of the Fourier series approaching the function it represents. One of the important tests for determining the convergence of the Fourier series is called the Dirichlet-Jordan test. Before moving on to the test, let us define partial sums of the Fourier series.

For some $n \in \mathbb{N}$, the partial sum of Fourier series of f is given by

$$s_n f(x) = \sum_{k=-n}^{n} c_k(f) e^{inx},$$

where $c_k(f)$ is k^{th} Fourier coefficient defined as in (1.2).

The Dirichlet-Jordan [79, p. 57] test for convergence of Fourier series is as given below:

Theorem P. If $f \in BV(\overline{\mathbb{T}})$ then at every point $x_0 \in \overline{\mathbb{T}}$, $s_n f(x)$ converges to $\frac{1}{2}\left[f(x+0) + f(x-0)\right]$. Furthermore, if f is continuous in $\overline{\mathbb{T}}$ then $s_n f(x)$ converges uniformly in $\overline{\mathbb{T}}$.

The test does not give any rate at which the partial sum of the Fourier series converges. The information on the rate of convergence of Fourier series is crucial for determining accuracy and understanding it helps in accessing how quickly the Fourier series approaches the original function. In 1971, Bojanić [8, p. 57] gave a quantitative version of the Dirichlet-Jordan test in terms of variations as follows.

Theorem Q. If $f \in BV([-\pi, \pi])$ and $g_x(t) = f(x+t) - f(x-t) - f(x+0) - f(x-0)$ then

$$|s_n f(x) - \frac{1}{2}\left[f(x+0) + f(x-0)\right]| \leq \frac{3}{n} \sum_{k=1}^{n} V\left(g_x, \left[0, \frac{\pi}{k}\right]\right),$$

where $V(g_x, [0,t])$ is variation of g_x on $[0,y]$, $y \in [0,\pi]$.

The above result of Bojanić was further generalized by Waterman [76, p. 52] in 1982 for Λ bounded variation as follows.

Theorem R. If $f \in \Lambda BV([-\pi, \pi])$, $g_x(t) = f(x+t) - f(x-t) - f(x+0) - f(x-0)$, $\frac{\lambda_k}{k}$ is non increasing and for fixed n, $H(t)$ is a continuously non increasing function on $(0, \pi]$ such that $H(t) = \frac{\lambda_k}{t}$; where $t = \frac{k\pi}{n+1}$ and $k = 1, 2, ..., (n+1)$ and $\frac{\pi}{n+1} = a_n < a_{n-1} < ... < a_0 = \pi$, then

$$|s_n f(x) - f(x)|$$
$$\leq \left(1 + \frac{2}{\pi}\right)\left[\frac{\lambda_{n+1}}{n+1}(V_\Lambda(g_x, [0,\pi])) + \frac{\pi}{n+1} \sum_{i=0}^{n-1} V_\Lambda(g_x, [0, a_i])(H(a_{i+1}) - H(a_i))\right],$$

where $V_\Lambda\left(g_x,[0,t]\right)$ is $\Lambda-$variation of g_x on $[0,y],\ y\in[0,\pi]$.

The conjugate Fourier series is given by

$$\sum_{n=-\infty}^{\infty}(-i)\text{sgn}(n)c_n(f)e^{inx}$$

and the partial sum of conjugate Fourier series of f is given by

$$\tilde{s}_nf(x)=\sum_{k=-n}^{n}(-i)\text{sgn}(k)c_k(f)e^{inx}.$$

In 1987, Mazhar and Al-Budaiwi [38, p. 178] obtained an estimate of the rate of convergence of conjugate Fourier series of functions of bounded variation.

Theorem S. If $f\in BV([-\pi,\pi])$ and $g_x(t)=f(x+t)-f(x-t)-f(x+0)-f(x-0)$, then

$$\left|\tilde{s}_nf(x)-\tilde{f}\left(x,\frac{\pi}{n}\right)\right|\leq\frac{3.3}{n}\sum_{k=1}^{n}V\left(g_x,\left[0,\frac{\pi}{k}\right]\right),$$

where $V\left(g_x,[0,t]\right)$ is variation of g_x on $[0,y],\ y\in[0,\pi]$ and

$$\tilde{f}(x;\epsilon)=\frac{-2}{\pi}\int_\epsilon^\pi\frac{g_x(t)}{2\tan(t/2)}dt.$$

For some $m,n\in\mathbb{N}$, the partial sum of double Fourier series of f is given by

$$s_{mn}f(x,y)=\sum_{l=-m}^{m}\sum_{k=-n}^{n}c_{lk}(f)e^{imx}e^{iny},$$

where $c_{lk}(f)$ is lk^{th} double Fourier coefficient defined as in (1.11). Also, let

$$S(f;x,y)=\frac{1}{4}[f(x+0,y+0)+f(x+0,y-0)+f(x-0,y+0)+f(x-0,y-0)]. \quad (1.18)$$

In 1906, Hardy [28] proved the following Dirichlet-Jordan test for double Fourier series of functions of bounded variation in Hardy sense.

Theorem T. If $f\in BV_H(\overline{\mathbb{T}}^2)$ then double Fourier series of f converges to $S(f;x,y)$ (1.18) at any point $(x,y)\in\overline{\mathbb{T}}^2$.

Note that the convergence mentioned above is of partial sum $s_{mn}f(x,y)$ in

Pringsheim's sense, i.e., when m and n tend to ∞, independently of one another. In 1992, Móricz [41, Theorem 3, p. 349] quantified Hardy's result and gave the following quantitative version of the Dirichlet-Jordan test for double Fourier series.

Theorem U. If $f \in BV_H(\overline{\mathbb{T}}^2)$ then

$$\begin{aligned}
&|s_{mn}f(x,y) - S(f;x,y)| \\
&\quad \le \frac{4(1+2/\pi+1/\pi^2)}{(m+1)(n+1)} \sum_{j=1}^{m}\sum_{k=1}^{n} V\left(\psi_{xy}, \left[0,\frac{\pi}{j}\right]\times\left[0,\frac{\pi}{k}\right]\right) \\
&\quad + \frac{2(1+1/\pi)}{m+1}\sum_{j=1}^{m} V\left(\psi_{xy}(.,0), \left[0,\frac{\pi}{j}\right]\right) \\
&\quad + \frac{2(1+1/\pi)}{n+1}\sum_{k=1}^{n} V\left(\psi_{xy}(0,.), \left[0,\frac{\pi}{k}\right]\right),
\end{aligned}$$

where $\psi(u,v) := \psi_{xy}(u,v)$

$$:= \begin{cases}
f(x+u,y+v)+f(x+u,y-v)+f(x-u,y+v) \\
\quad +f(x-u,y-v)-S(f;x,y) \text{ if } u,v>0 \\
f(x+0,y+v)+f(x-0,y+v)+f(x+0,y-v) \\
\quad +f(x-0,y-v)-S(f;x,y) \text{ if } u=0,\ v>0 \\
f(x+u,y+0)+f(x-u,y+0)+f(x+u,y-0) \\
\quad +f(x-u,y-0)-S(f;x,y) \text{ if } u>0,\ v=0 \\
0 \text{ if } u=v=0.
\end{cases}$$

and $S(f;x,y)$ is as defined in (1.18).

The above result is a two-variable extension of the result due to Bojanić Theorem Q.

1.5 Rate of convergence of rational Fourier series

For $n \in \mathbb{N}$ and $x \in [-\pi, \pi]$, the partial sum of rational Fourier series of f is given by

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k)\phi_k(e^{ix}),$$

where $\hat{f}(k)$ is k^{th} rational Fourier coefficient (1.8) of f.

The conjugate rational Fourier series is given by

$$\sum_{n=-\infty}^{\infty} (-i)\mathrm{sgn}(n)\hat{f}(n)\phi_n(e^{ix}).$$

and the partial sum of conjugate Fourier series of f is given by

$$\tilde{S}_n f(x) = \sum_{k=-n}^{n} (-i)\mathrm{sgn}(k)\hat{f}(k)\phi_k(e^{ix}).$$

Here, if we take $\alpha_k = 0$, for all $k \in \mathbb{N}$ in (1.5), then the conjugate rational Fourier series reduces to the conjugate Fourier series.

The analogous Dirichlet-Jordan test for rational Fourier series was obtained by Džrbašyan [19, Theorem 1, p. 26] when the rational Fourier series was introduced. Tan and Qian [60, Theorem 2.4, p. 545] obtained the rate of convergence of rational and conjugate rational Fourier series for continuous functions of bounded variation as follows:

Theorem V. If $f \in BV([0, 2\pi])$ is continuous then

$$|S_n f(x) - f(x)| \leq \frac{3(1+r)}{1-r} \sum_{k=1}^{n} V\left(\phi_x, \left[\frac{-\pi}{l}, \frac{\pi}{k}\right]\right),$$

$$\left|\tilde{S}_n f(x) - \tilde{f}\left(x; \frac{\pi}{n}\right)\right| \leq \frac{3(1+r)}{1-r} \sum_{k=1}^{n} V\left(\phi_x, \left[\frac{-\pi}{l}, \frac{\pi}{k}\right]\right),$$

where $\phi_x(t) = f(x) - f(x-t)$ and

$$\tilde{f}(x;h) = \frac{1}{\pi}\int_{h\le|t|\le\pi} \frac{f(x-t)}{2\tan(t/2)}dt.$$

The above result generalizes Theorem Q and Theorem S for rational Fourier series for continuous functions.

In Chapter 4, the result of Theorem V (p. 31) is generalized and rate of convergence of rational and conjugate rational Fourier series for functions of $\Lambda-$ bounded variation is established. Thus, an analogous result to Waterman's Theorem R (p. 28) is obtained for rational and conjugate rational Fourier series. Furthermore, Theorem R (p. 28) is extended for two variable continuous functions of $\Lambda-$ variation in Hardy sense for double rational Fourier series, thus generalizing the result of Móricz, Theorem U (p. 30).

1.6 Convergence and integrability of trigonometric and double trigonometric series

The study of convergence of trigonometric series provides insights into how well a trigonometric series approaches a given function and is closely linked to orthogonality properties of sine and cosine or exponential functions. Conditions for trigonometric series' integrability provide insights into series convergence and the behaviour of the converging functions in a broader sense. The study of partial differential equations often involves the use of trigonometric series. Convergence properties are crucial for solutions represented by these series.

Definition 1.6.1. Let $x \in \overline{\mathbb{T}}$ and $\{c(n)\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers, then (complex) trigonometric series is defined as

$$\sum_{|n|<\infty} c(n)e^{inx}. \tag{1.19}$$

It is worth noting that if the coefficients of the trigonometric series are Fourier coefficients of some function, then the trigonometric series becomes the

Fourier series of that function. Understanding convergence and integrability properties is essential for analysing functions in context of Fourier series and other orthogonal functions.

The study of trigonometric series is conducted with conditions on the sequence of complex numbers, $\{c(n)\}_{n\in\mathbb{Z}}$. The concept of bounded variation for sequences [7, p. 3] is defined as follows:

Definition 1.6.2. A sequence of complex numbers $\{c(n)\}_{n\in\mathbb{Z}}$ is said to be of bounded variation, written as $\{c(n)\}_{n\in\mathbb{Z}} \in \mathcal{BV}$, if

$$\sum_{n\in\mathbb{Z}} |c(n)| < \infty.$$

In 1954, Ul'yanov [64] obtained the following significant results for sine and cosine series.

Theorem W. Let $f(x) = \sum_{n=1}^{\infty} a_n \cos nx$ and $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$. If $\{a_n\}_{n=1}^{\infty}$ is a null sequence of bounded variation then $f, g \in L^p[0, 2\pi)$ for any $0 < p < 1$.

In 1980, the concept of bounded variation of higher order for sequences was defined by Garrett et. al. [23, Definition 1.3, p. 424].

Definition 1.6.3. A sequence of complex numbers $\{c(n)\}_{n=-\infty}^{\infty}$ is said to be of bounded variation of order m (denoted by $\mathcal{BV}^m$), $m \in \mathbb{N}$ if

$$\sum_{n=-\infty}^{\infty} |\Delta^m c(n)| < \infty,$$

where $\Delta^m c(n) = \Delta^{m-1}c(n) - \Delta^{m-1}c(n+1)$ and $\Delta^0 c(n) = c(n)$.

The following inclusion relation [23, p. 424] holds:

$$\mathcal{BV}^m \subset \mathcal{BV}^{m+1}.$$

The above inclusion is strict and can be shown using the example discussed by Garrett et. al. [23, p. 424].

In 1984, Stanojevic [58, p. 371] generalized Ul'yanov's result for trigonometric series by considering the condition of bounded variation of higher order for the coefficients of the series as follows:

Theorem X. If for some $m \in \mathbb{N}$, a complex sequence $\{c(n)\}_{n\in\mathbb{Z}} \in \mathcal{BV}^m$ then the trigonometric series (1.19)

(i) converges pointwise to some function $f(x)$ for every $x \in \mathbb{T} \setminus \{0\}$.

(ii) converges in $L^p(\mathbb{T})$-metric to f for any $0 < p < \frac{1}{m}$.

In 1988, Moricz [40] studied convergence and integrability of double trigonometric series by defining bounded variation for double sequences.

Definition 1.6.4. The double trigonometric series is defined as

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c(j,k) e^{ijx} e^{iky}, \tag{1.20}$$

where $\{c(j,k) : -\infty < j,k < \infty\}$ is a double sequence of complex numbers and $(x,y) \in \mathbb{T}^2$.

Let the rectangular partial sums of double trigonometric series (1.20) be given by

$$S_{JK}(x,y) = \sum_{|j|\le J} \sum_{|k|\le K} c(j,k) e^{ijx} e^{iky}. \tag{1.21}$$

The series (1.21) is said to converge in Pringsheim's sense to $f(x,y)$ if $S_{JK}(x,y) \to f(x,y)$ as $\min(J,K) \to \infty$. In addition, if the row series $\sum_{j=-\infty}^{\infty} c(j,k)e^{ijx}e^{iky}$ converges for each fixed value of k and the column series $\sum_{k=-\infty}^{\infty} c(j,k)e^{ijx}e^{iky}$ converges for each fixed value of j then the double trigonometric series (1.20) is said to converge regularly [29] to $f(x,y)$.

Definition 1.6.5. A double sequence of complex numbers $\{c(j,k)\}_{(j,k)\in\mathbb{Z}^2}$ is said to be of bounded variation (denoted by $\mathcal{BV}_2$) if

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta c(j,k)| < \infty,$$

where $\Delta c(j,k) = c(j,k) - c(j+1,k) - c(j,k+1) + c(j+1,k+1)$.

Moricz [40] obtained the following result for double trigonometric series.

Theorem Y. If a double complex sequence $\{c(j,k)\}_{(j,k)\in\mathbb{Z}^2} \in \mathcal{BV}_2$ and $c(j,k) \to 0$ as $\max(|j|,|k|) \to \infty$, then the double trigonometric series (1.20)

(i) converges regularly to some function $f(x,y)$ for every $(x,y) \in (\mathbb{T}\backslash\{0\})^2$.

(ii) converges in $L^p(\mathbb{T}^2)$-metric to f for any $0 < p < 1$ when $\min(j,k) \to \infty$.

In 1998, Chen and Wu [12, p. 395] defined the notion of bounded variation of higher order for double sequences.

Definition 1.6.6. A double sequence of complex numbers $\{c(j,k)\}_{(j,k)\in\mathbb{Z}^2}$ is said to be of bounded variation of order m (denoted by $\mathcal{BV}_2^m$) if $c(j,k) \to 0$ as $\max(|j|,|k|) \to \infty$ and for $m \in \mathbb{N}$,

$$\lim_{|k|\to\infty} \sum_{j=-\infty}^{\infty} |\Delta_{m0} c(j,k)| = 0,$$

$$\lim_{|j|\to\infty} \sum_{k=-\infty}^{\infty} |\Delta_{0m} c(j,k)| = 0$$

$$\text{and} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{mm} c(j,k)| < \infty;$$

where

$$\Delta_{00} c(j,k) = c(j,k),$$

$$\Delta_{mn} c(j,k) = \Delta_{m-1,n} c(j,k) - \Delta_{m-1,n} c(j+1,k) \ (m \geq 1)$$

and

$$\Delta_{mn} c(j,k) = \Delta_{m,n-1} c(j,k) - \Delta_{m,n-1} c(j,k+1) \ (n \geq 1).$$

Here,

$$\Delta_{mn} c(j,k) = \sum_{p=0}^{m} \sum_{q=0}^{n} (-1)^{p+q} \binom{m}{p} \binom{n}{q} c(j+p,k+q). \tag{1.22}$$

Kaur et al. [32, Theorem 3.1, p. 272] studied results related to convergence and integrability of double trigonometric series where the double sequence of coefficients is of double bounded variation of order m.

Theorem Z. If for some $m \in \mathbb{N}$, a double complex sequence $\{c(j,k)\}_{(j,k)\in\mathbb{Z}^2} \in \mathcal{BV}_2^m$, then the double trigonometric series (1.20)

i) converges regularly to some function $f(x, y)$ for every $(x, y) \in (\mathbb{T}\setminus\{0\})^2$.

ii) converges in $L^p(\mathbb{T}^2)$-metric to f for any $0 < p < \frac{1}{m}$ when $\min(j, k) \to \infty$.

In Chapter 5, the rational trigonometric series with the orthogonal system as the Takenaka Malmquist system having fixed poles is considered. The result related to convergence and integrability similar to Theorem X (p. 34) is obtained for this rational trigonometric series. Furthermore, this result is extended by considering double rational trigonometric series and thus the result similar to Theorem Z (p. 35) for double rational trigonometric series is obtained.