Chapter 2

Order of magnitude of rational Fourier coefficients

The Riemann Lebesgue Lemma is insufficient in providing the order of magnitude of classical Fourier coefficients and the order of Fourier coefficients holds significant mathematical importance for understanding the behaviour of the function. Thus, the study of the order of magnitude of Fourier coefficients for functions of bounded variation has been the subject of extensive research. Recently, in 2013, Tan and Zhou [62] gave the Riemann Lebesgue Lemma type result for rational Fourier series. Also, they estimated the order of magnitude of rational Fourier coefficients for the function of $\Phi - \Lambda-$ bounded variation, which was analogous to the result for Fourier coefficients due to Schramm and Waterman (Theorem C on p. 8).

The order of magnitude of classical Fourier coefficients varies depending on the class of functions of generalized bounded variation considered. Thus, it is interesting to note the difference when these results are obtained for rational Fourier coefficients. In this chapter, the results for order of magnitude of rational Fourier coefficients for the classes of functions of generalized variations like Akhobadze's $B\Lambda(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ class (Definition 1.1.7 on p. 10)) and Vyas' $\Lambda BV(p(n) \uparrow p, \varphi, \overline{\mathbb{T}})$ class (Definition 1.1.6 on p. 8) are obtained, using the technique given by Tan and Zhou [62]. Also, the order of rational Fourier coefficients for functions in $Lip(\alpha, p)[0, 2\pi]$ class (Definition 1.1.1 on p. 2) is estimated.

Recall that we have assumed that the parameters α_k, defined in the rational orthogonal system (1.5), satisfies the condition (1.6) and r is as defined in (1.6). Also, the notation $\mathbb{T} = [0, 2\pi)$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ will be used throughout this chapter.

Theorem 2.0.1. *If $f \in Lip(p;\beta)(\overline{\mathbb{T}}),\ p \geq 1,\ \beta \in (0,1]$ and $m \in \mathbb{Z}^*$ then*

$$\hat{f}(m) = O\left(\frac{1}{|m|^\beta}\right).$$

Proof. There exists $h_1 \in [0, 2\pi]$ such that $\theta_{|m|}(x + h_1) - \theta_{|m|}(x) = \pi$ and we also have $|h_1| \leq \frac{(1+r)\pi}{(1-r)|m|}$ which implies $|\rho_{|m|}(x + h_1) - \rho_{|m|}(x)| \leq \frac{r(1+r)^2\pi}{|m|(1-r)^4}$. Then for $m \in \mathbb{Z}^*$,we have

$$\begin{aligned}
\hat{f}(m) &= \frac{1}{2\pi}\int_{\overline{\mathbb{T}}} f(x)\overline{\phi_m(e^{ix})}dx \\
&= \frac{1}{2\pi}\int_{\overline{\mathbb{T}}} f(x)\rho_{|m|}(x)e^{-i\ sgn(m)\ \theta_{|m|}(x)}dx \\
&= \frac{1}{4\pi}\int_{\overline{\mathbb{T}}} [f(x)\rho_{|m|}(x) - f(x+h_1)\rho_{|m|}(x+h_1)]e^{-i\ sgn(m)\ \theta_{|m|}(x)}dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\hat{f}(m)| &\leq \frac{1}{4\pi}\int_{\overline{\mathbb{T}}} |f(x+h_1)\rho_{|m|}(x+h_1) - f(x)\rho_{|m|}(x)|dx \\
&\leq \frac{1}{4\pi}\left[\int_{\overline{\mathbb{T}}} f(x+h_1) - f(x)||\rho_{|m|}(x+h_1)|dx \right. \\
&\qquad \left. + \int_{\overline{\mathbb{T}}} |f(x)|\{|\rho_{|m|}(x+h_1) - \rho_{|m|}(x)|\}dx\right] \\
&\leq \frac{1}{4\pi}\left[\sqrt{\frac{1+r}{1-r}}\int_{\overline{\mathbb{T}}} |f(x+h_1) - f(x)|dx \right. \\
&\qquad \left. + \int_{\overline{\mathbb{T}}} |f(x)|dx \left\{\frac{r(1+r)^2\pi}{|m|(1-r)^4}\right\}\right].
\end{aligned}$$

Using Jensen's inequality, we get

$$\begin{aligned}
|\hat{f}(m)|^p &\leq \frac{1}{4\pi}\left[\left(\frac{1+r}{1-r}\right)^{p/2}\int_{\overline{\mathbb{T}}} |f(x+h_1) - f(x)|^p dx + \frac{(1+r)^{2p}(r\pi)^p}{|m|^p(1-r)^{4p}}||f||_p^p\right] \\
&\leq \frac{1}{4\pi}\left[\left(\frac{2\pi(1+r)}{1-r}\right)^{p/2}\left\{\omega^{(p)}\left(f; \frac{(1+r)\pi}{(1-r)|m|}\right)\right\}^p\right.
\end{aligned}$$

$$+\frac{(1+r)^{2p}(r\pi)^p}{|m|^p(1-r)^{4p}}||f||_p^p\Bigg]$$

$$\leq \frac{1}{4\pi}\left[C\left(\frac{2\pi(1+r)}{1-r}\right)^{p/2}\left(\frac{(1+r)\pi}{(1-r)|m|}\right)^{\beta p}+\frac{(1+r)^{2p}(r\pi)^p}{|m|^p(1-r)^{4p}}||f||_p^p\right]$$

where C is some positive constant. Thus,

$$|\hat{f}(m)|^p = O\left(\frac{1}{|m|^{\beta p}}+\frac{1}{|m|^p}\right).$$

Hence, the result follows. □

Remark 1. The above result of the order of rational Fourier coefficients is analogous to the classical result of the Fourier coefficients, Theorem A (p. 2).

Lemma 2.0.2. *$\Lambda BV(p(n)\uparrow p,\varphi,\overline{\mathbb{T}})$ and $\Lambda BV\left(p(n)\uparrow p,\frac{2(1+r)}{1-r}\varphi,\overline{\mathbb{T}}\right)$ coincides for $1\leq p\leq\infty$.*

The result can be proved by following similar steps as in [4, Lemma 3].

Theorem 2.0.3. *If $f\in\Lambda BV(p(n)\uparrow p,\varphi,\overline{\mathbb{T}})$, $1\leq p\leq\infty$ and $n\in\mathbb{Z}^*$, then*

$$\hat{f}(n)=O\left(\frac{1}{\left(\sum_{j=1}^{2|n|}\frac{1}{\lambda_j}\right)^{\frac{1}{p(\tau(|n|))}}}+\frac{1}{|n|}\right),$$

where

$$\tau(n)=\min\{k:k\in\mathbb{N},\varphi(k)\geq n\}, n\geq 1. \tag{2.1}$$

Proof. By the property of $\theta_{|n|}(x)$ and following similar steps as in [62, Theorem 2.2], we get, for an integer $j\in[0,2|n|]$, there exists an increasing sequence $x_j\in[0,2\pi]$ such that $\theta_{|n|}(x+x_j)-\theta_{|n|}(x)=j\pi$ and by the mean value theorem,

$$|\rho_{|n|}(x+x_j)-\rho_{|n|}(x+x_{j-1})|\leq\frac{r(1+r)^{1/2}}{(1-r)^{5/2}}(x_j-x_{j-1})\leq\frac{r(1+r)^{3/2}\pi}{(1-r)^{7/2}|n|} \tag{2.2}$$

as

$$\frac{(1-r)\pi}{(1+r)|n|}\leq x_j-x_{j-1}\leq\frac{(1+r)\pi}{(1-r)|n|}.$$

Therefore, for $n \in \mathbb{Z}^*$,

$$\hat{f}(n) = \frac{(-1)^j}{2\pi} \int_{\overline{\mathbb{T}}} f(x+x_j)\rho_{|n|}(x+x_j)e^{-i \ \mathrm{sgn}(n) \ \theta_{|n|}(x)} \ dx.$$

By using $\rho_{|n|}(x-x_j) \leq \sqrt{\frac{1+r}{1-r}}$ and (2.2), we get,

$$\begin{aligned}|\hat{f}(n)| &\leq \frac{1}{4\pi}\int_{\overline{\mathbb{T}}} |f(x+x_j)\rho_{|n|}(x+x_j) - f(x+x_{j-1})\rho_{|n|}(x+x_{j-1})|dx \\ &\leq c_1 \int_{\overline{\mathbb{T}}} |f(x+x_j) - f(x+x_{j-1})|dx + \frac{c_2}{|n|},\end{aligned}$$

where $c_1 = \frac{1}{4\pi}\left(\frac{1+r}{1-r}\right)^{1/2}$ and $c_2 = \frac{r(1+r)^{3/2}||f||_1}{4(1-r)^{7/2}}$.

Dividing both the sides by λ_j and summing from $j = 1$ to $2|n|$, we get

$$\left(|\hat{f}(n)| - \frac{c_2}{|n|}\right)\sum_{j=1}^{2|n|}\frac{1}{\lambda_j} \leq c_1 \int_{\overline{\mathbb{T}}} \sum_{j=1}^{2|n|} \frac{|f(x+x_j) - f(x+x_{j-1})|}{\lambda_j} dx.$$

By applying Holder's inequality on the right side of the above inequality, we get

$$\left(|\hat{f}(n)| - \frac{c_2}{|n|}\right)\sum_{j=1}^{2|n|}\frac{1}{\lambda_j}$$

$$\leq c_1 \int_{\overline{\mathbb{T}}} \left(\sum_{j=1}^{2|n|} \frac{|f(x+x_j) - f(x+x_{j-1})|^{p(\tau(|n|))}}{\lambda_j}\right)^{\frac{1}{p(\tau(|n|))}} \left(\sum_{j=1}^{2|n|}\frac{1}{\lambda_j}\right)^{\frac{1}{q(\tau(n))}} dx,$$

where $\frac{1}{p(\tau(|n|))} + \frac{1}{q(\tau(|n|))} = 1$ and $p(\tau(|n|))$ is as given in (2.1).
Hence,

$$|\hat{f}(n)| \leq \frac{2\pi c_1 V_{\Lambda_{p(n)}}(f, \frac{2(1+r)}{1-r}\varphi, \overline{\mathbb{T}})}{\left(\sum_{j=1}^{2|n|}\frac{1}{\lambda_j}\right)^{\frac{1}{p(\tau(|n|))}}} + \frac{c_2}{|n|}.$$

Thus, the proof follows from Lemma 2.0.2. □

Remark 2. If $\alpha_k = 0$, $\forall k \in \mathbb{N}$, then in the above result, $r = 0$, thus only the first term remains as the second term is multiple of r. Hence, we get the analogous result of Fourier coefficients, similar to Theorem D on p. 9.

Theorem 2.0.4. *If $f \in B\Lambda(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}})$ and $n \in \mathbb{Z}^*$, then*

$$\hat{f}(n) = O\left(\frac{1}{\mid n \mid^{\frac{1}{p(\tau(|n|))}}}\right),$$

where

$$\tau(n) = \min\left\{k : k \in \mathbb{N}, \varphi(k) \geq n\left(\frac{1+r}{1-r}\right)\right\}, n \geq 1. \tag{2.3}$$

Proof. There exists $h_j \in [0, 2\pi]$ such that $\theta_{|n|}(x + h_j) - \theta_{|n|}(x) = j\pi$ for $j = 1, 2$ and $h_1 < h_2$. By the mean value theorem, we get

$$\mid \rho_{|n|}(x + h_2) - \rho_{|n|}(x + h_1) \mid \leq \frac{r(1+r)^{1/2}}{(1-r)^{5/2}}(h_2 - h_1) \leq \frac{r(1+r)^{3/2}\pi}{(1-r)^{7/2} \mid n \mid}$$

as

$$\frac{(1-r)\pi}{(1+r) \mid n \mid} \leq h_2 - h_1 \leq \frac{(1+r)\pi}{(1-r) \mid n \mid}.$$

Therefore, for $n \in \mathbb{Z}^*$ and by following similar steps as in [62, Theorem 2.1], we get

$$\begin{aligned}
\mid \hat{f}(n) \mid &\leq \frac{1}{4\pi}\int_{\overline{\mathbb{T}}} \mid f(x + h_2)\rho_{|n|}(x + h_2) - f(x + h_1)\rho_{|n|}(x + h_1) \mid dx \\
&\leq c_1 \int_{\overline{\mathbb{T}}} \mid f(x + h_2) - f(x + h_1) \mid dx + \frac{c_2}{\mid n \mid},
\end{aligned}$$

where $c_1 = \frac{1}{4\pi}\left(\frac{1+r}{1-r}\right)^{1/2}$ and $c_2 = \frac{r(1+r)^{3/2}||f||_1}{4(1-r)^{7/2}}$.

Let $h_2 - h_1 = h$, $\frac{1}{p(\tau(|n|))} + \frac{1}{q(\tau(|n|))} = 1$ and $p(\tau(\mid n \mid))$ be as given in (2.3), then by Hölder's inequality, we have

$$\begin{aligned}
\mid \hat{f}(n) \mid &\leq c_1 \int_{\overline{\mathbb{T}}} h^{-\frac{1}{p(\tau(|n|))}} \mid f(x + h_2) - f(x + h_1) \mid h^{\frac{1}{p(\tau(|n|))}} dx + \frac{c_2}{\mid n \mid} \\
&\leq c_1 \left\{h^{-1}\int_{\overline{\mathbb{T}}} \mid f(x + h_2) - f(x + h_1) \mid^{p(\tau(|n|))} dx\right\}^{\frac{1}{p(\tau(|n|))}} \\
&\quad \times \left\{\int_{\overline{\mathbb{T}}} h^{\frac{q(\tau(|n|))}{p(\tau(|n|))}} dx\right\}^{\frac{1}{q(\tau(|n))}} + \frac{c_2}{\mid n \mid} \\
&\leq c_1\Lambda(f, p(\mid n \mid) \uparrow \infty, \varphi, \overline{\mathbb{T}})(2\pi)^{\frac{1}{q(\tau(|n|))}} h^{\frac{1}{p(\tau(|n|))}} + \frac{c_2}{\mid n \mid}
\end{aligned}$$

$$= O\left(\frac{1}{| n |^{\frac{1}{p(\tau(|n))}}}\right).$$

Hence, the result is obtained. □

Remark 3. If $\alpha_k = 0, \ \forall k \in \mathbb{N}$, then in the above result, by putting $r = 0$, we get the analogous result for Fourier coefficients, Theorem E on p. 10.