Synopsis of the Thesis entitled

Mathematical Study of Compact Stars in General Relativity

submitted by

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1 Introduction

Albert Einstein developed the gravitational theory known as general theory of relativity in 1915. Gravity was viewed in classical mechanics as a force, that is operated between two objects having mass. However, by adding the idea of spacetime curvature, Einstein's general relativity offers a fresh perspective on gravity. According to the general theory of relativity the presence of matter curves up the geometry of associated spacetime, where the metric is described by the Riemannian metric

$$ds^2 = g_{ij}dx^i dx^j.$$

The metric coefficients g_{ij}, is known as fundamental tensor, plays an important role in the formation of Einstein's field equations. Einstein's field equations are described as

$$R_{ij} - \frac{1}{2}Rg_{ij} = -\frac{8\pi G}{c^4}T_{ij},$$

where, $i,\ j = 0,\ 1,\ 2,\ 3.$ R_{ij} denotes Ricci tensor, R denotes Ricci scalar and T_{ij} is the energy-momentum tensor that contains information of physical properties of spacetime. R_{ij} and R have the following expressions:

$$R_{ij} = \frac{\partial}{\partial x^j}\Gamma^k_{ik} + \Gamma^l_{ik}\Gamma^k_{lj} + \Gamma^l_{ij}\Gamma^k_{lk} - \frac{\partial}{\partial x^k}\Gamma^k_{ij} \tag{1}$$

where

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(g_{lj,i} + g_{li,j} - g_{ij,l})$$

are components of Christoffel symbol of second kind and

$$R = g^{ij}R_{ij}.$$

For static spherically symmetric spacetime metric,

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2}$$

Einstein's field equation leads to four second order non-linear ordinary differential equations, out of which three equations are independent. Due to the non-linear nature of Einstein's field equations, it is difficult to obtain a closed-form (exact) solution of Einstein's field equations. Exact solution plays a significant role in understanding the properties of compact stars. The energy momentum tensor for anisotropic fluid distribution can be defined as

$$T_{ij} = (\rho + p_\perp)u_i u_j + p_\perp g_{ij} + (p_r - p_\perp)\chi_i\chi_j, \tag{3}$$

where ρ is the matter density, p_r is the radial pressure, $p_\perp$ is the tangential pressure, u^i is the four-velocity of the fluid and χ^i is a unit spacelike four-vector along the radial direction so that $u^i u_i = -1, \chi^i \chi_j = 1$ and $u^i \chi_j = 0$, for spacetime metric (2) and energy-momentum tensor (3), the Einstein's field equations takes the form

$$8\pi\rho = \frac{1 - e^{-\lambda}}{r^2} + \frac{e^{-\lambda}\lambda'}{r}, \tag{4}$$

$$8\pi p_r = \frac{e^{-\lambda}\nu'}{r} + \frac{e^{-\lambda} - 1}{r^2}, \tag{5}$$

$$8\pi p_\perp = e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu' - \lambda'}{2r}\right), \tag{6}$$

$$8\pi\Delta = 8\pi p_r - 8\pi p_\perp, \tag{7}$$

where primes denote differentiation with respect to r. The system of equation (4-7) governs the behavior of the gravitational field for anisotropic fluid distribution.

1.1 Elementary Criteria for Physical Acceptability

The non linearity of Einstein's field equations makes challenging to find exact solutions. Presently, a number of distinct exact solutions are available in the literature (for e.g.[8]). Delgaty and Lake[5] studied 127 solutions of Einstien's field equations out of which 16 only satisfy physical plausibility conditions and only 9 solutions satisfy causality conditions. The following conditions must be satisfied for a solution to be physically acceptable [[9], [3], [6] and [10]].

(a) Regularity Conditions:

(i) The solution should be free from the physical and geometric singularities. i.e. $e^{\lambda(r)} > 0, e^{\nu(r)} > 0$ in the range $0 \leq r \leq a$, where a is the radius of the stellar object.
(ii) The radial and transverse pressures and density of the distribution should be non negative, i.e. $\rho(r) \geq 0,\ p_r(r) \geq 0,\ p_\perp(r) \geq 0, \quad \text{for } 0 \leq r \leq a.$
(iii) Radial pressure p_r should vanish at the boundary r = a i.e. $p_r(r = a) = 0$.

(b) Behavior of measure of anisotropy:

Pressure anisotropy $\Delta = p_r - p_\perp$ should vanish at the centre, i.e. $\Delta(0) = 0$.

(c) Energy Condition:

Each of the energy conditions, namely Weak Energy Condition (WEC), Null Energy Condition (NEC) and Strong Energy Condition (SEC) must be satisfied for an anisotropic fluid sphere which are as follows

$$(i) NEC : \rho > 0, \tag{8}$$

$$(ii)WEC : \rho - p_r > 0, \rho + p_\perp \geq 0, \tag{9}$$

$$(iii)SEC : \rho - p_r - 2p_\perp \geq 0, \tag{10}$$

(d) Monotone decrease of physical parameters:

The pressure and density should be maximum at the center of the star and monotonically decrease towards the boundary. Mathematically, this means,

$\frac{d\rho}{dr} = 0, \; \frac{dp_r}{dr} = 0, \; \frac{dp_\perp}{dr} = 0, \qquad \text{for } r = 0$

and

$\frac{d^2\rho}{dr^2} < 0, \; \frac{d^2p_r}{dr^2} < 0, \; \frac{d^2p_\perp}{dr^2} < 0, \qquad \text{for } r = 0$

so that,

$\frac{d\rho}{dr} < 0, \; \frac{dp_r}{dr} < 0, \; \frac{dp_\perp}{dr} < 0, \qquad \text{for } 0 \leq r \leq a$

(e) Equation of state:

The ratio of pressure to density, $\frac{p_r}{\rho}, \frac{p_\perp}{\rho}$, should be monotonically decreasing towards the boundary. i.e. $\frac{d}{dr}\left(\frac{p_r}{\rho}\right) = 0$ and $\frac{d}{dr}\left(\frac{p_\perp}{\rho}\right) = 0$ at r = 0 and $\frac{d^2}{dr^2}\left(\frac{p_r}{\rho}\right) < 0$ at r = 0 and $\frac{d^2}{dr^2}\left(\frac{p_\perp}{\rho}\right) < 0$ at r = 0.

(f) Charge Distribution:

The electric field intensity E should be such that $E(0) = 0$ and monotonically increasing towards the boundary. i.e. $\frac{dE}{dr} > 0$, for $0 < r < a$.

(g) Mass to Radius Ratio:

The allowable mass to radius ratio is given by $\frac{M}{a} \leq \frac{4}{9}$. According to [3], the mass radius relation must satisfy the inequality, $\frac{M}{a} \leq \frac{4}{9}$. But when we introduce the electric field inside the matter distribution, it modifies this upper limit as proposed by Buchdahl's. Later, [1] (upper limit of $\frac{M_{ch}}{a}$) and [2] (lower limit of $\frac{M_{ch}}{a}$) give the modified mass-radius limit in the presence of electric charge inside the matter distribution, which can be described as follows:

$$\frac{Q^2(18a^2 + Q^2)}{2a^2(12a^2 + Q^2)} \leq \frac{M_{ch}}{a} \leq \frac{2a^2 + 3Q^2 + 2a\sqrt{a^2 + 3Q^2}}{9a^2},$$

where, M_{ch} is the total mass of the compact object for the charged perfect fluid matter distribution.

(h) Gravitational Redshift and Surface Redshift:

The surface redshift z_s can be obtained as,

$$z_s = \left(1 - 2\frac{m(r)}{r}\right)^{-\frac{1}{2}} - 1. \tag{11}$$

The gravitational redshift z should be monotonically decreasing towards the boundary of the star. The central redshift z_c and boundary redshift z_a must be positive and finite. That is,

$$z_c = e^{-\nu/2} - 1 > 0, at \;\; r = 0$$

and

$$z_a = e^{-\nu/2} - 1 > 0, at \;\; r = a.$$

(i) Causality condition:
$0 \leq \frac{dp_r}{d\rho} \leq 1, \;\; 0 \leq \frac{dp_\perp}{d\rho} \leq 1, \quad$ for $0 \leq r \leq a$.
The values for the radial speed of sound waves $\frac{dp_r}{d\rho}$ denoted as ν_r^2 and transverse speed of sound waves $\frac{dp_\perp}{d\rho}$ denoted as ν_t^2. These velocities are in the range of 0 and 1.

(j) Matching Conditions:

(I) For uncharged matter distribution, the interior solution obtained should match continuously with Schwarzschild exterior metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{12}$$

at the boundary of the star $r = a$. This gives $e^{\nu(a)} = e^{-\lambda(a)} = 1 - \frac{2M}{a}$.

(II) For a charged matter distribution, the interior metric should match with the Reissner-Nordström exterior spacetime

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{13}$$

This leads to across the boundary r = a of the star. This gives
$e^{\nu(a)} = e^{-\lambda(a)} = 1 - \frac{2M}{a} + \frac{Q^2}{a^2}$.
Further the static steller configuration must be stable.

2 Outline of the thesis:

Chapter:1

This chapter contains the introduction and derivation of Einstein's field equations for static spherically symmetric anisotropic matter distribution. Schwarzschild interior and exterior solutions are also discussed in this chapter.

Chapter:2

A class of new solutions for Einstein's field equations, by choosing the ansatz $e^{\lambda(r)} = \frac{1+k\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}$ for static spherically symmetric spacetime metric (2), are obtained under Karmarkar condition. Which says

$$R_{1414}R_{2323} = R_{1212}R_{3434} + R_{1224}R_{1334}, \tag{14}$$

substituting the components of Riemann curvature tensor R_{ijkl} in (14). The karmarkar condition takes the form

$$\frac{\nu''}{\nu'} + \frac{\nu'}{2} = \frac{\lambda' e^{\lambda}}{2\left(e^{\lambda}-1\right)}. \tag{15}$$

The general solution of equation (15) is given by

$$e^{\nu} = \left[A + B\int \sqrt{(e^{\lambda(r)}-1)}dr\right]^2, \tag{16}$$

where A and B are constants of integration.

The anisotropy takes the form

$$8\pi\sqrt{3}S = -\frac{\nu' e^{-\lambda}}{4}\left[\frac{2}{r} - \frac{\lambda'}{e^{\lambda}-1}\right]\left[\frac{\nu' e^{\nu}}{2rB^2} - 1\right]. \tag{17}$$

In the case of isotropic distribution of matter, we have $S = 0$ which leads to either $\frac{2}{r} - \frac{\lambda'}{e^{\lambda}-1} = 0$ or $\frac{\nu' e^{\nu}}{2rB^2} - 1 = 0$. The former case leads to Schwarzschild [12] exterior solution and the latter gives the solution given by Kohler and Chao [7]. It is found that a number of pulsars like 4U 1820-30, PSR J1903+327, 4U 1608-52, Vela X-1, PSR J1614-2230, Cen X-3 can be accommodated in this model. We have displayed the nature of physical parameters and energy conditions throughout the distribution using numerical and graphical methods for a particular pulsar 4U 1820-30 and found that the solution satisfies all physical requirements.

Chapter:3

In this chapter we have studied a new class of interior solutions that are singularity-free and useful for describing anisotropic compact star objects with spherically symmetric matter distribution. We have considered metric potential g_{rr} as $B_0^2(r) = \frac{1}{(1-\frac{r^2}{R^2})^n}$,where $n > 2$. The various physical characteristics of the model are specifically examined for the pulsar PSRJ1903+327 with estimated data. According to analysis, every condition need for a physically admissible star is satisfied. Further the stability of the model has been examined. Numerous physical characteristics are also highlighted in a graphical form.

To develop a physically reasonable model of the stellar configuration, we assume that the metric potential g_{rr} is expressed

$$B_0^2(r) = \frac{1}{(1-\frac{r^2}{R^2})^n}, \tag{18}$$

where $n > 0$ is any real number. By selecting this metric potential, the function $B_0^2(r)$ is guaranteed to be finite, continuous and well-defined within the range of stellar interiors. Also $B_0^2(r) = 1$ for $r = 0$ ensures that it is finite at the center. Again, the metric is regular at the center since $(B_0^2(r))'_{r=0} = 0$. We have generalized the work of Das et.al.[4], where authors have developed model for steller configuration by considering n = 4. It is observed that all the physical quantities are well behaved up to n = 70.

Chapter:4

In this chapter new exact solutions of Einstein's field equations for charged stellar models have been derived by choosing ansatz $e^{\lambda} = 1+\frac{r^2}{R^2}$ and considering linear equation of state for radial pressure $P_r = A\rho - B$, where A and B are constants. The expression of charge is consider as

$$E^2 = \frac{\alpha\frac{r^2}{R^2}}{R^2(1+\frac{r^2}{R^2})^2}, \tag{19}$$

The physical acceptability conditions of the model are investigated, and the model is compatible with several compact star candidates like 4U 1820-30, PSR J1903+327, EXO 1785-248, LMC X-4, SMC X-4, Cen X-3. A noteworthy feature of the model is that it satisfies all the conditions needed for a physically acceptable model. It is obseved that when $\alpha = 0$. i.e. in the case of uncharged matter distribution the model reduces to the model studied by Thomas and Pandya [13].

Chapter:5

A new form of linear equation of state relating radial pressure and density on Finch Skea spacetime have been considered in this chapter. The solution of field equations have been obtained and expression of density, radial pressure and tangential pressure have been calculated. The interior spacetime metric is matched with Schwarzschild exterior spacetime metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{20}$$

and the values of constants of integration and mass have been obtained. It is observed that the total mass of steller configuration is one fourth of radious. Further, all the physical plusibility conditions are satisfied.

Chapter:6

Nasheeha et.al.[11] studied that models of steller configuration by considering metric potential $g_{rr} = \frac{1+ar^2}{1+(a-b)r^2}$ and equation of state

$$p_r = \tau\rho^{(1+\frac{1}{p})} + \eta\rho - \omega, \tag{21}$$

where τ, η, ω and p are real constants. It is noted that the metric potential g_{tt} and many physical entities are not well-behaved in the case of a = b. We consider metric potential $g_{rr} = 1 + ar^2$ which is particular case of $g_{rr} = \frac{1+ar^2}{1+(a-b)r^2}$ when a = b. If p=1 in equation (21), then it becomes a quadratic equation of state. If $\tau = 0$ in equation (21), then it becomes a linear equation of state. If $\eta = 0$, in equation (21), then it becomes polytrope with polytropic index p. If $p = \frac{-1}{2}, \omega = 0$ and $\tau = -\alpha$, in equation (21), then it becomes chaplygin equation of state. If $p = -2$, then it becomes a color-flavor-locked (CFL) equation of state. The physical viability of models is tested for strange star candidate 4U 1820 - 30 having mass $M = 1.58M_\odot$ and radius R = 9.1 km. All the models are found to be physically plausible.

3 Paper Presented in Conference:

- I have presented the paper "Relativistic stars under Karmarkar condition on pseudo-spheroidal spacetime" in ICMMAAC-21 held at JECRC University, Jaipur (Raj.), India, in the duration of 5-7 August 2021

- I have presented the paper "New charged anisotropic solution on paraboloidal spacetime" in ICMTA-2022 held at the Department of Mathematics, SRM Institute of Science and Technology, in the duration of 23-25 March 2022.
- I have presented the paper "A new model of compact star suitable with pulsar 4U 1820-30" held at the Department of Mathematics, Veer Narmad South Gujarat University, duration of 4-5 March 2023.

4 *List of Research Paper published in Journals:*

- Ratanpal, B S and Thomas, V O and Patel, Rinkal, Compact relativistic stars under Karmarkar condition,*New Astronomy*, **100** (2023) 101970.
- Ratanpal, B S and Patel, Rinkal, Anisotropic approach: compact star as a generalized model, *Astrophysics and Space Science*, **368** (2023) 21.
- Patel, Rinkal and Ratanpal, B S and Pandya, D M, New charged anisotropic solution on paraboloidal spacetime, *Astrophysics and Space Science*, **368** (2023) 58.

5 *List of Research articles Communicated in Journals:*

- I have Communicated "Anisotropic star with linear equation of state" in Chinese Journal of Physics.
- I have Communicated "A various equation of state for Anisotropic models for a compact star" in The European Physical Journal-Plus.

Some of References are as follows.

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